Bipanconnectivity and bipancyclicity in k-ary n-cubes

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Abstract

In this paper we give precise solutions to problems posed by Wang, An, Pan, Wang and Qu and by Hsieh, Lin and Huang. In particular, we show that Q_n^k is bipanconnected and edge-bipancyclic, when $k \geq 3$ and $n \geq 2$, and we also show that when k is odd, Q_n^k is m-panconnected, for $m = \frac{n(k-1)+2k-6}{2}$, and $(k-1)$-pancyclic (these bounds are optimal). We introduce a path-shortening technique, called progressive shortening, and strengthen existing results, showing that when paths are formed using progressive shortening then these paths can be efficiently constructed and used to solve a problem relating to the distributed simulation of linear arrays and cycles in a parallel machine whose interconnection network is Q_n^k, even in the presence of a faulty processor.
keywords: interconnection networks; k-ary n-cubes; bipanconnectivity; bipancyclicity.

1 Introduction

The choice of interconnection network is crucial in the design of a distributed-memory multiprocessor. As to which network is chosen depends upon a number of factors relating to the topological, algorithmic and communication properties of the network and the types of problems to which the resulting computer is to be applied. One of the most popular interconnection networks is undoubtedly the n-dimensional hypercube Q_n. Some of its pleasing properties, with regard to parallel computation, include: it is vertex- and edge-symmetric; it is Hamiltonian; it has diameter n; it has a recursive decomposition; and it contains, or 'nearly' contains (as subgraphs), almost all interconnection networks currently vogue in parallel computing (see [18] for these results and more on the hypercube). Some of the commercial machines whose underlying topology is based on the hypercube are the Cosmic Cube [23], the Ametek S/14 [2], the iPSC [10, 11], the Ncube [7, 11] and the CM-200 [8].

However, every vertex of Q_n has degree n, and, consequently, as n increases so does the degree of every vertex. High degree vertices in interconnection networks can lead to technological problems in parallel computers whose underlying topology is that of the said interconnection network. One method of circumventing this problem, so as to still retain a 'hypercube-like' interconnection network, is to build parallel computers so that the underlying topology is the k-ary n-cube Q_n^k. The k-ary n-cube Q_n^k is similar in essence to the hypercube, but by a judicious choice of k and n we can include a large number of vertices yet keep the degree of each vertex fixed. For example, the hypercube Q_{12} has 4096 vertices and every vertex has degree 12. However, Q_3^{16} has 4096 vertices and every vertex has degree 6. Of course, one usually loses out in some other respect (for example, in terms of diameter) but often this loss is not too catastrophic. The k-ary n-cube Q_n^k has not been investigated to the same extent as the hypercube, but it is known to have the following properties (amongst many others): it is vertex- and edge-symmetric [3]; it is Hamiltonian [4, 6]; it has diameter $n\lfloor k/2 \rfloor$ [4, 6]; it has a recursive decomposition; and it contains many important interconnection networks such as cycles (of certain lengths) [3], meshes (of certain dimensions) [4] and even hypercubes (of certain dimensions) [6]. Machines whose underlying topology is based on a k-ary n-cube include the Mosaic [24], the iWARP [5], the J-machine [21], the Cray T3D [16] and the Cray T3E [1].

Of interest to us in this paper are the different paths and cycles embedded within k-ary n-cubes. Path and cycle networks are fundamental in parallel computing; not only is there a multitude of algorithms specifically designed for linear arrays of processors and cycles of processors but paths and cycles appear as data structures in many more algorithms for parallel machines whose processors are inter-connected in a variety of topologies. For example, having a collection of processors connected in a cycle means that all-to-all message passing can be undertaken by "daisy-chaining" messages around the cycle. Of particular interest to us are questions relating to Hamiltonicity, pancyclicity, panconnectivity, bipancyclicity and bipanconnectivity (these concepts are defined in the next section). These properties can be described as 'strong Hamiltonicity' properties and their existence in an interconnection network enables a much higher degree of flexibility with regard to the simulation of linear arrays of processors or cycles of processors.

The notions in the preceding paragraph have been investigated in the context of a number of interconnection networks: for example, in crossed cubes [12, 31], Möbius cubes [14], augmented cubes [20], alternating group graphs [9], star graphs [29], bubble-sort graphs [17], and in hypercubes and hypercube-like networks [13, 19, 22, 25, 26, 28, 30]. As regards k-ary n-cubes, these notions have been considered in [15, 27]. In particular, it was proven in [27]: that Q_2^k is almost-Hamiltonian connected, bipanconnected and bipancyclic; that Q_n^k is almost-Hamiltonian connected, for any k; and that Q_n^k is Hamiltonian-connected, for odd k. Recently, it has been proven in [15] that Q_n^3 is edge-pancyclic. It was posed as an open problem in [27] as to whether their results on bipanconnectivity and bipancyclicity for Q_2^k could be extended to Q_n^k, for arbitrary n, and it was posed as an open problem in [15] as to whether their results on panconnectivity and pancyclicity could be extended to Q_n^k, for arbitrary k. In this paper, we provide precise answers to both these questions. In addition, we show that when k is odd, Q_n^k is m-panconnected, for $m = \frac{n(k-1)+2k-6}{2}$, and $(k-1)$-pancyclic (these bounds are optimal). We also strengthen the results in [15, 27] by introducing a path-shortening technique, called progressive shortening, and show that the construction of paths using this technique enables us to efficiently construct paths in a distributed fashion and so solve a problem relating to the distributed simulation of linear arrays and cycles in a parallel machine whose interconnection network is Q_n^k, even in the presence of a faulty processor (even in Q_2^k, the solution to this problem is not possible using the paths constructed in [27]).

In the next section, we present some basic definitions and results, before improving the constructions from [27] in Q_2^k in Section 3. In Section 4, we look at the general case when k is even, and in Section 5 when k is odd. We outline our application in Section 6 before presenting our conclusions in

Section 7.

2 Basic definitions and results

The *k-ary n-cube* Q_n^k, for $k \geq 3$ and $n \geq 2$, has vertex set $\{0, 1, \ldots, k-1\}^n$, and there is an edge $((u_{n-1}, u_{n-2}, \ldots, u_0), (v_{n-1}, v_{n-2}, \ldots, v_0))$ if, and only if, there exists $d \in \{0, 1, \ldots, n-1\}$ such that $min\{|u_d - v_d|, k - |u_d - v_d|\} = 1$, and $u_i = v_i$, for every $i \in \{0, 1, \ldots, n-1\} \setminus \{d\}$. Many structural properties of k-ary n-cubes are known, but of particular relevance for us is that a k-ary n-cube is *vertex-symmetric*; that is, given any two distinct vertices v and v' of Q_n^k, there is an automorphism of Q_n^k mapping v to v'. Throughout, we assume that addition on tuple elements is modulo k. The *parity* of a vertex $(v_{n-1}, v_{n-2}, \ldots, v_0)$ of Q_n^k is defined to be $(\Sigma_{i=0}^{n-1} v_i \bmod 2)$ (note that if k is even then every edge of Q_n^k joins an even parity vertex to an odd parity vertex).

An index $d \in \{0, 1, \ldots, n-1\}$ is often referred to as a *dimension*. We can *partition Q_n^k over dimension d* by fixing the dth element of any vertex tuple at some value a, for every $a \in \{0, 1, \ldots, k-1\}$. This results in k copies $Q_d(0), Q_d(1), \ldots, Q_d(k-1)$ of Q_{n-1}^k (with $Q_d(a)$ obtained to fixing the dth element at a), with corresponding vertices in $Q_d(0), Q_d(1), \ldots, Q_d(k-1)$ joined in a cycle of length k (in dimension d). Such a partition proves to be extremely useful.

It has long been known that every k-ary n-cube Q_n^k is *Hamiltonian*, *i.e.*, contains a cycle passing through every vertex exactly once. A *Hamiltonian path* in a graph is a path joining two vertices on which every vertex of the graph appears exactly once, and a graph is *Hamiltonian-connected* if there is a Hamiltonian path joining any pair of distinct vertices. Note that any (non-trivial) bipartite graph cannot be Hamiltonian-connected, though there might exist *almost-Hamiltonian paths*, *i.e.*, paths joining pairs of distinct vertices upon which all but one of the vertices of the graph appear; a solitary vertex not appearing on an almost-Hamiltonian path is called the *residual vertex*. Irrespective of whether a graph is bipartite or not, we say that a graph is *almost-Hamiltonian-connected* if there is a Hamiltonian path or an almost-Hamiltonian path joining any pair of distinct vertices. It is proven in [27] that every k-ary n-cube Q_n^k is almost-Hamiltonian-connected, and that if k is odd then Q_n^k is Hamiltonian-connected.

We say that a graph G on n vertices is *pancyclic* (resp. *m-pancyclic*) if it contains a cycle of every possible length between 3 and n (resp. m and n). The graph G is *almost-pancyclic* if it contains a cycle of every possible length between 4 and n, and *bipancyclic* if it contains a cycle of every possible even length between 4 and n (the definition of bipancyclicity is intended primarily for bipartite graphs but can be applied to any graph). A graph G is *edge-bipancyclic* if there exists an edge e of G such that e lies on a cycle of every even length between 4 and n. The graph G is *panconnected* (resp. *m-panconnected*) if for any pair of distinct vertices u and v, there is a path joining u and v of every length between $d(u,v)$ (resp. $m > d(u,v)$) and $n-1$, where $d(u,v)$ is the length of a minimal length path in G joining u and v. The graph G is *bipanconnected* if for any pair of distinct vertices u and v, there is a path joining u and v of every length from $\{l : l = d(u,v) + 2i,$ where $0 \leq i \leq \frac{n-d(u,v)}{2}\}$. It is proven in [27] that Q_2^k is bipanconnected and (edge-) bipancyclic; however, as to whether Q_n^k, for $n \geq 3$, is bipanconnected or bipancyclic was left as an open question. However, in relation to this question, it was proven in [15] that Q_n^3 is edge-pancyclic, for all $n \geq 2$.

Our final definition concerns the alteration of paths joining two distinct vertices in Q_n^k. Let u and v be distinct vertices of Q_n^k and let ρ be a path joining u to v of length m, where $m - d(u,v)$ is even. Suppose that there are paths $\rho_{d(u,v)}, \rho_{d(u,v)+2}, \ldots, \rho_m = \rho$ such that:

- the path ρ_i joins u and v and is of length i, for each $i = d(u,v)$, $d(u,v)+2, \ldots, m$
- for each $i = d(u,v), d(u,v)+2, \ldots, m-1$, the path ρ_{i+1} is of the form

  $$u = u_0, u_1, \ldots, u_{i+1} = v$$

  with ρ_i of the form

  $$u = u_0, u_1, \ldots, u_j, u_{j+3}, u_{j+4}, \ldots, u_{i+1} = v,$$

  for some $j \in \{0, 1, \ldots, i-2\}$.

Then we say that ρ can be *progressively shortened* to obtain paths of all lengths from $\{l : l = d(u,v), d(u,v)+2, \ldots, m\}$. As we shall see, it will be crucial that our paths can be progressively shortened.

3 Existing bipanconnectivity results

The result from [27] that Q_2^k is bipanconnected (irrespective of whether k is odd or even) is important to our forthcoming results (as the base case of inductions). However, we need to refine the proof from [27] that Q_2^k is bipanconnected in order to obtain a stronger result, involving progressive shortening, and so that we can apply this stronger result later. We remark that it is also crucial that any residual vertex is as stated in Proposition 1. Our stronger result is as follows.

Proposition 1 *Let $k \geq 3$ and let u and v be distinct vertices of Q_2^k.*

1. *If $k + d(u,v)$ is odd then there exists a Hamiltonian path joining u and v such that this path can be progressively shortened to obtain paths of all lengths from $\{d(u,v) + 2i : 0 \leq i \leq \frac{(k^2-1-d(u,v))}{2}\}$.*

2. *If $k+d(u,v)$ is even then there exists an almost-Hamiltonian path joining u and v such that the residual vertex is adjacent to v and such that this path can be progressively shortened to obtain paths of all lengths from $\{d(u,v) + 2i : 0 \leq i \leq \frac{(k^2-2-d(u,v))}{2}\}$.*

In particular, Q_2^k is bipannconnected.

Before we prove Proposition 1, let us illustrate why the proof from [27] that Q_2^k is panconnected will not suffice. Consider Case (a) of Fig. 2 in [27] (in this case, k is even). We have reproduced this figure in Fig. 1(a). The authors claim (in a statement prior to Theorem 3) that the almost-Hamiltonian path joining u and v can be shortened to a path of length $d(u,v)$ so that paths of lengths $d(u,v), d(u,v)+2, \ldots, k^2-2$ are obtained, and this is indeed the case. However, regard the path from u to v as a curve on the plane and close this curve as shown in Fig. 1 with the dotted line. No matter how we *progressively* shorten the almost-Hamiltonian path, the residual vertex (shaded in grey) must lie inside the closed curve, and hence we cannot shorten the almost-Hamiltonian path to a path of length $d(u,v)$ (as any such path must lie within the top-left shaded grid). We have corrected this deficiency in Fig. 1(b).

Similarly, the cases in Fig. 2(c) and Fig. 3(d) in [27] are deficient in the same way, and have been reproduced in Fig. 2(a,c). These deficiencies are corrected in Fig. 2(b,d). Thus, Proposition 1 follows (as all other cases in [27] are such that the paths can be progressively shortened).

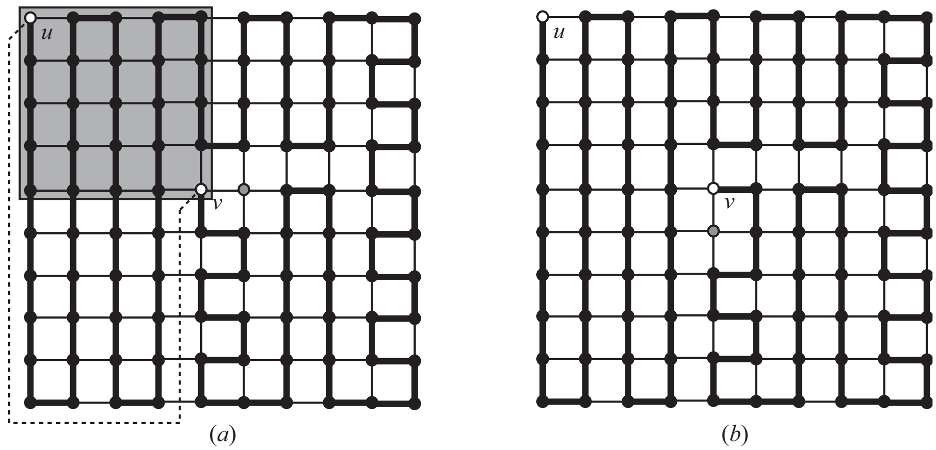

Figure 1. Case (a) of Fig. 2 of [27] and its correction.

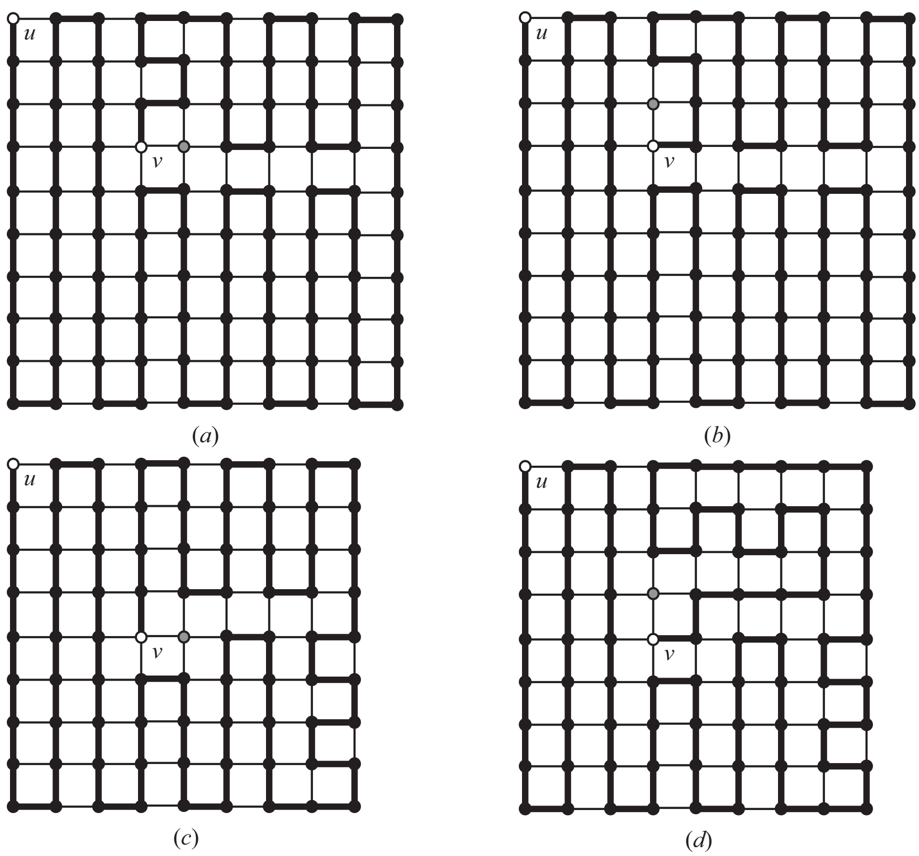

Figure 2. Other cases from [27] and their corrections.

4 The general case when k is even

We begin by examining whether Q_n^k is bipanconnected or not when k is even (we reiterate that Q_n^k is bipartite when k is even). As remarked earlier, this question was posed as an open problem by Wang, An, Pan, Wang and Qu in [27]. We answer this question precisely.

Theorem 2 *Let $k \geq 4$ and $n \geq 2$, with k even, and let u and v be distinct vertices of Q_n^k.*

1. *If $d(u,v)$ is odd then there exists a Hamiltonian path joining u and v such that this path can be progressively shortened to obtain paths of all odd lengths between $d(u,v)$ and $k^n - 1$, inclusive.*

2. *If $d(u,v)$ is even then there exists an almost-Hamiltonian path joining u and v such that the residual vertex is adjacent to v and such that this path can be progressively shortened to obtain paths of all even lengths between $d(u,v)$ and $k^n - 2$, inclusive.*

In particular, Q_n^k is bipannconnected.

Proof The vertex-symmetry of Q_n^k means that, w.l.o.g., we may suppose that $u = (0,0,\ldots,0)$ and $v = (v_{n-1}, v_{n-2}, v_{n-3}, \ldots, v_0)$, where $v_i \leq \frac{k}{2}$, for $i = 0,1,\ldots,n-1$, and where $v \neq (v_{n-1}, 0, \ldots, 0)$. For brevity, denote v_{n-1} as a.

Let $u^i = (i,0,0,\ldots,0)$, for $0 \leq i \leq k-1$; hence, $u = u^0$ and $v \neq u^a$. Partition Q_n^k over dimension $n-1$ to obtain $Q_n^k(0), Q_n^k(1), \ldots, Q_n^k(k-1)$. We proceed by induction on n. There are two cases, according to whether $d(u^a, v)$ is odd or even.

Case (i) $d(u^a,v)$ is odd.

So, by the induction hypothesis applied to $Q_n^k(a)$, there exists a Hamiltonian path ρ_a from u^a to v in $Q_n^k(a)$ which can be progressively shortened to obtain paths of all odd lengths between $d(u^a,v) = d(u,v) - a$ and $k^{n-1} - 1$, inclusive. Note that if the parity of v is even (resp. odd) then a is odd (resp. even).

Denote the vertex $(i, v_{n-2}, v_{n-3}, \ldots, v_0)$ as v^i, for $i \in \{0,1,\ldots,k-1\}$; so, $v = v^a$. For each $i \in \{0,1,\ldots,k-1\} \setminus \{a\}$, let $\rho_i \in Q_n^k(i)$ be obtained from ρ_a by setting the first component of every vertex of ρ_a at i. Note that corresponding vertices of the paths $\rho_0, \rho_1, \ldots, \rho_{k-1}$ induce cycles of length k in Q_n^k, *e.g.*, $u^0, u^1, \ldots, u^{k-1}, u^0$ is a cycle of length k, as is $v^0, v^1, \ldots, v^{k-1}, v^0$.

In particular, the edges of these induced cycles and the edges of the paths $\rho_0, \rho_1, \ldots, \rho_{k-1}$ yield a $k \times k^{n-1}$ grid, with rows $1, 2, \ldots, k$ and columns $1, 2, \ldots, m$, where $m = k^{n-1}$, with 'wrap-around' column edges. Refer to the vertices by their row-column co-ordinates in this grid; so, for example, u is the vertex $(1, 1)$ and v is the vertex $(a + 1, m)$.

Sub-case $(i.a)$ Suppose that a is even (and so v lies on odd row $a + 1$). Consider the path ρ from u to v defined as:

$$
\begin{aligned}
&(1,1),(2,1),\ldots,(k,1),(k,2),(k-1,2),\ldots,(1,2),\\
&\quad(1,3),(2,3),\ldots,(k,3),(k,4),(k-1,4),\ldots,(1,4),\\
&\quad\ldots,(1,m-3),(2,m-3),\ldots,(k,m-3),(k,m-2),\\
&\quad(k-1,m-2),\ldots,(1,m-2),(1,m-1),(k,m-1),\\
&\quad(k-1,m-1),\ldots,(a+2,m-1),(a+2,m),(a+3,m),\ldots,\\
&\quad(k-1,m),(k,m),(1,m),(2,m),(2,m-1),(3,m-1),(3,m),\\
&\quad(4,m),(4,m-1),\ldots,(a,m),(a,m-1),(a+1,m-1),(a+1,m).
\end{aligned}
$$

The path ρ is Hamiltonian and can be visualized as in Fig. $3(a)$. Furthermore, it can trivially be progressively shortened to obtain paths of all odd lengths between $k^{n-1} - 1 + a$ and $k^n - 1$ (inclusive), and so that the path of length $k^{n-1} - 1 + a$ is the path ρ_0 in $Q_n^k(0)$, from u to v^0, extended with the path in column m of length a to vertex v. By above, the path ρ^0 can be progressively shortened to obtain paths of all odd lengths between $d(u, v^0) = d(u, v) - a$ and $k^{n-1} - 1$, and we obtain the required result.

Sub-case $(i.b)$ Suppose that a is odd (and so v lies on even row $a + 1 \geq 2$). Consider the path ρ from u to v defined as:

$$
\begin{aligned}
&(1,1),(2,1),\ldots,(k,1),(k,2),(k-1,2),\ldots,(1,2),\\
&\quad(1,3),(2,3),\ldots,(k,3),(k,4),(k-1,4),\ldots,(1,4),\\
&\quad\ldots,(1,m-3),(2,m-3),\ldots,(k,m-3),(k,m-2),\\
&\quad(k-1,m-2),\ldots,(1,m-2),(1,m-1),(k,m-1),\\
&\quad(k-1,m-1),\ldots,(a+2,m-1),(a+2,m),\\
&\quad(a+3,m),\ldots,(k-1,m),(k,m),(1,m),(2,m),\\
&\quad(2,m-1),(3,m-1),(3,m),(4,m),(4,m-1),\ldots,\\
&\quad(a,m-1),(a,m),(a+1,m)
\end{aligned}
$$

(note that the vertex $(a + 1, m - 1)$ does not appear on ρ).

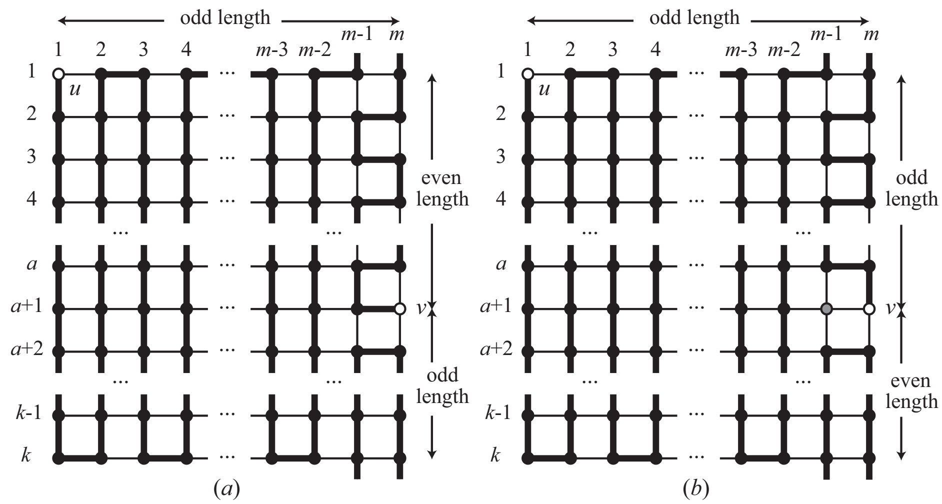

Figure 3. The different cases when $d(u^a, v)$ is odd.

The path ρ is almost-Hamiltonian and can be visualized as in Fig. 3(b). Furthermore, it can trivially be progressively shortened to obtain paths of all even lengths between $k^{n-1} - 1 + a$ and $k^n - 2$, and so that the path of length $k^{n-1} - 1 + a$ is the path ρ_0 in $Q^k_n(0)$, from u to v^0, extended with the path in column m of length a from v^0 to v. By above, the path ρ^0 can be progressively shortened to obtain paths of all odd lengths between $d(u, v^0)$ and $k^{n-1} - 1$. As $d(u, v) = d(u, v^0) + a$ and the vertex $(a + 1, m - 1)$ is adjacent to v, we obtain the required result.

Case (ii) $d(u^a, v)$ is even.

So, by the induction hypothesis applied to $Q^k_n(a)$, there exists an almost-Hamiltonian path ρ_a from u^a to v in $Q^k_n(a)$ which can be progressively shortened to obtain paths of all even lengths between $d(u^a, v) = d(u, v) - a$ and $k^{n-1} - 2$, and so that the residual vertex of the almost-Hamiltonian path ρ_a is adjacent to v. Note that if the parity of v is even (resp. odd) then a is even (resp. odd).

For each $i \in \{0, 1, \ldots, k - 1\} \setminus \{a\}$, let $\rho_i \in Q^k_n(i)$ be obtained from ρ_a by setting the first component of every vertex of ρ_a at i. As was the case in Case (i), corresponding vertices of the paths $\rho_0, \rho_1, \ldots, \rho_{k-1}$ induce cycles of length k in Q^k_n. In particular, the edges of these induced cycles and the edges of the paths $\rho_0, \rho_1, \ldots, \rho_{k-1}$ yield a $k \times (k^{n-1} - 1)$ grid, with rows $1, 2, \ldots, k$ and columns $1, 2, \ldots, m - 1$, where $m = k^{n-1}$, with 'wrap-around' column edges. Furthermore, if we denote the residual vertex of ρ_i

in $Q^k_n(i)$ by r^i then there is an edge (v^i, r^i) in Q^k_n, for $i = 0, 1, \ldots, k-1$; moreover, $r^0, r^1, \ldots, r^{k-1}, r^0$ is a cycle (this is why we focus on the adjacency relationship between the residual vertex and the vertex v, as in the statement of the result). Thus, we have a $k \times m$ grid with 'wrap-around' column edges, just as we had in Case (i); as before, we refer to the vertices as row-column pairs.

Sub-case $(ii.a)$ Suppose that a is even (and so v lies on odd row $a+1 \geq 1$ and on column $m-1$). Consider the path ρ from u to v defined as:

$$\begin{aligned}&(1,1),(2,1),\ldots,(k,1),(k,2),(k-1,2),\ldots,(1,2),\\&\quad(1,3),(2,3),\ldots,(k,3),(k,4),(k-1,4),\ldots,(1,4),\\&\quad\ldots,(1,m-3),(2,m-3),\ldots,(k,m-3),(k,m-2),\\&\quad(k,m-1),(k,m),(k-1,m),\ldots,(a+2,m),(a+2,m-1),\\&\quad(a+3,m-1),\ldots,(k-1,m-1),(k-1,m-2),\\&\quad(k-2,m-2),\ldots,(1,m-2),(1,m-1),(1,m),\\&\quad(2,m),(2,m-1),(3,m-1),(3,m),(4,m),\\&\quad(4,m-1),\ldots,(a,m),(a,m-1),(a+1,m-1)\end{aligned}$$

(note that the vertex $(a+1,m)$ does not appear on ρ). The path ρ is almost-Hamiltonian and can be visualized as in Fig. 4(a). Furthermore, it can trivially be progressively shortened to obtain paths of all even lengths between $k^{n-1}-2+a$ and k^n-2, and so that the path of length $k^{n-1}-2+a$ is the path ρ_0 in $Q^k_n(0)$, from u to v^0, extended with the path in column $m-1$ of length a from v^0 to v. By above, the path ρ^0 can be progressively shortened to obtain paths of all even lengths between $d(u,v^0)$ and $k^{n-1}-2$. As $d(u,v) = d(u,v^0)+a$ and the vertex $(a+1,m)$ is adjacent to v, we obtain the required result.

Sub-case $(ii.b)$ Suppose that a is odd (and so v lies on even row $a+1 \geq 2$ and on column $m-1$). Consider the path ρ from u to v defined as:

$$\begin{aligned}&(1,1),(2,1),\ldots,(k,1),(k,2),(k-1,2),\ldots,(1,2),\\&\quad(1,3),(2,3),\ldots,(k,3),(k,4),(k-1,4),\ldots,(1,4),\\&\quad\ldots,(1,m-3),(2,m-3),\ldots,(k,m-3),(k,m-2),\\&\quad(k-1,m-2),\ldots,(1,m-2),(1,m-1),(1,m),(2,m),\\&\quad(2,m-1),(3,m-1),(3,m),(4,m),(4,m-1),\ldots,\\&\quad(a,m-1),(a,m),(a+1,m),(a+2,m),\ldots,(k-1,m),(k,m),\\&\quad(k,m-1),(k-1,m-1),\ldots,(a+2,m-1),(a+1,m-1).\end{aligned}$$

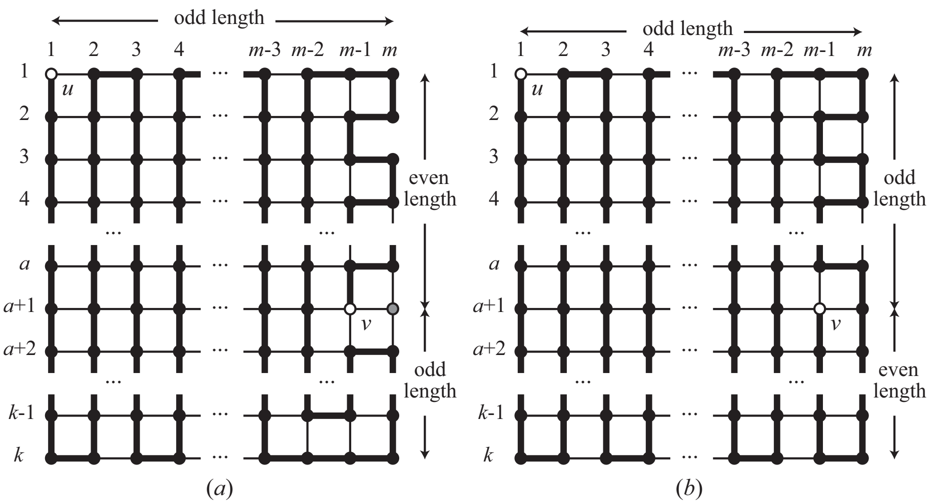

Figure 4. The different cases when $d(u^a, v)$ is even.

The path ρ is Hamiltonian and can be visualized as in Fig. 4(b). Furthermore, it can trivially be progressively shortened to obtain paths of all odd lengths between $k^{n-1} - 2 + a$ and $k^n - 1$, and so that the path of length $k^{n-1} - 2 + a$ is the path ρ_0 in $Q_n^k(0)$, from u to v^0, extended with the path in column $m - 1$ of length a from v^0 to v. By above, the path ρ^0 can be progressively shortened to obtain paths of all even lengths between $d(u, v^0) = d(u, v) - a$ and $k^{n-1} - 2$; thus, we obtain the required result.

All that remains is to deal with the base case of the induction. However, the base case is handled by Proposition 1. $\square$

The following is an immediate corollary of Theorem 2.

Corollary 3 *Let $k \geq 4$ and $n \geq 2$, with k even. Q_n^k is edge-bipancyclic.*

5 The general case when k is odd

We now examine whether Q_n^k is bipanconnected when k is odd. As remarked earlier, this question was posed as an open problem by Wang, An, Pan, Wang and Qu in [27]. We answer this question precisely; in fact, we prove even more as we shall see later.

Theorem 4 *Let $k \geq 3$ and $n \geq 2$, with k odd, and let u and v be distinct vertices of Q_n^k.*

1. *If $d(u,v)$ is even then there exists a Hamiltonian path joining u and v such that this path can be progressively shortened to obtain paths of all even lengths between $d(u,v)$ and $k^n - 1$, inclusive.*

2. *If $d(u,v)$ is odd then there exists an almost-Hamiltonian path joining u and v such that the residual vertex is adjacent to v and such that this path can be progressively shortened to obtain paths of all odd lengths between $d(u,v)$ and $k^n - 2$, inclusive.*

In particular, Q_n^k is bipannconnected.

Proof The proof is very similar in structure to that of Theorem 2 and we adopt the exact same notation as in that proof. Again, we proceed by induction on n and there are two cases, according to whether $d(u^a, v)$ is odd or even.

Case (i) $d(u^a, v)$ is even.
So, by the induction hypothesis, there exists a Hamiltonian path ρ_a from u^a to v in $Q_n^k(a)$ which can be progressively shortened to obtain paths of all even lengths between $d(u^a, v) = d(u, v) - a$ and $k^{n-1} - 1$, inclusive. As in the proof Theorem 2, the paths $\rho_0, \rho_1, \ldots, \rho_{k-1}$ yield a $k \times k^{n-1}$ grid, with rows $1, 2, \ldots, k$ and columns $1, 2, \ldots, m$, where $m = k^{n-1}$, with 'wrap-around' column edges.
Sub-case $(i.a)$ Suppose that a is even (and so v lies on odd row $a+1 \geq 1$ and on column m). Consider the path ρ from u to v defined as:

$$\begin{aligned}
&(1,1), (2,1), \ldots, (k,1), (k,2), (k-1,2), \ldots, (1,2),\\
&\quad (1,3), (2,3), \ldots, (k,3), (k,4), (k-1,4), \ldots, (1,4), \ldots,\\
&\quad (k, m-3), (k-1, m-3), \ldots, (1, m-3), (1, m-2),\\
&\quad (2, m-2), \ldots, (k, m-2), (k, m-1), (k, m), (k-1, m),\\
&\quad (k-1, m-1), (k-2, m-1), (k-2, m), \ldots, (a+2, m),\\
&\quad (a+2, m-1), (a+1, m-1), (a, m-1), \ldots, (1, m-1),\\
&\quad (1, m), (2, m), \ldots, (a+1, m).
\end{aligned}$$

The path ρ is Hamiltonian and can be visualized as in Fig. 5(a). Similarly to as in the proof of Theorem 2, ρ can be progressively shortened to obtain paths of all even lengths between $d(u,v)$ and $k^n - 1$.

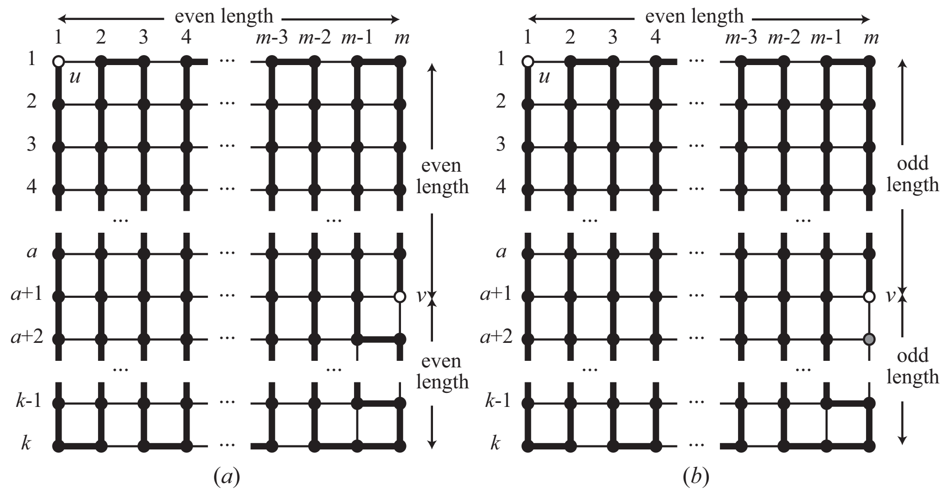

Figure 5. The different cases when $d(u^a, v)$ is even.

Sub-case $(i.b)$ Suppose that a is odd (and so v lies on even row $a+1 \geq 2$ and on column m). Consider the path ρ from u to v defined as:

$$\begin{aligned}&(1,1),(2,1),\ldots,(k,1),(k,2),(k-1,2),\ldots,(1,2),\\&\quad(1,3),(2,3),\ldots,(k,3),(k,4),(k-1,4),\ldots,(1,4),\ldots,\\&\quad(k,m-3),(k-1,m-3),\ldots,(1,m-3),(1,m-2),(2,m-2),\\&\quad\ldots,(k,m-2),(k,m-1),(k,m),(k-1,m),\\&\quad(k-1,m-1),(k-2,m-1),(k-2,m),(k-3,m),\\&\quad(k-3,m-1),\ldots,(a+2,m-1),(a+1,m-1),(a,m-1),\\&\quad\ldots,(1,m-1),(1,m),(2,m),\ldots,(a+1,m)\end{aligned}$$

(note that the vertex $(a+2, m)$ does not appear on ρ). The path ρ is almost-Hamiltonian and can be visualized as in Fig. 5(b). Similarly to as in the proof of Theorem 2, ρ can be progressively shortened to obtain paths of all odd lengths between $d(u, v)$ and $k^n - 2$.

Case (ii) $d(u^a, v)$ is odd.

So, by the induction hypothesis, there exists an almost-Hamiltonian path ρ_a from u^a to v in $Q_n^k(a)$ which can be progressively shortened to obtain paths of all odd lengths between $d(u^a, v) = d(u, v) - a$ and $k^{n-1} - 2$, and so that the residual vertex of the almost-Hamiltonian path ρ_a is adjacent to v. As in the proof Theorem 2, the paths $\rho_0, \rho_1, \ldots, \rho_{k-1}$ and the residual vertices yield a $k \times k^{n-1}$ grid, with rows $1, 2, \ldots, k$ and columns $1, 2, \ldots, m$, where $m = k^{n-1}$, with 'wrap-around' column edges.

Sub-case $(ii.a)$ Suppose that a is odd (and so v lies on even row $a + 1 \geq 2$ and on column $m - 1$). Consider the path ρ from u to v defined as:

$$
\begin{aligned}
&(1,1),(2,1),\ldots,(k,1),(k,2),(k-1,2),\ldots,(1,2),\\
&\quad(1,3),(2,3),\ldots,(k,3),(k,4),(k-1,4),\ldots,(1,4),\\
&\quad\ldots,(k,m-3),(k,m-2),(k-1,m-2),(k-1,m-3),\ldots,\\
&\quad(a+2,m-3),(a+2,m-2),(a+1,m-2),(a+1,m-3),\\
&\quad(a,m-3),(a,m-2),\ldots,(4,m-2),(4,m-3),(3,m-3),\\
&\quad(3,m-2),(2,m-2),(2,m-3),(1,m-3),(1,m-2),(1,m-1),\\
&\quad(k,m-1),(k-1,m-1),\ldots,(a+2,m-1),(a+2,m),(a+3,m),\\
&\quad\ldots,(k,m),(1,m),(2,m),(2,m-1),(3,m-1),(3,m),(4,m),\\
&\quad(4,m-1),\ldots,(a,m-1),(a,m),(a+1,m),(a+1,m-1).
\end{aligned}
$$

The path ρ is Hamiltonian and can be visualized as in Fig. 6(a). Similarly to as in the proof of Theorem 2, ρ can be progressively shortened to obtain paths of all even lengths between $d(u, v)$ and $k^n - 1$.

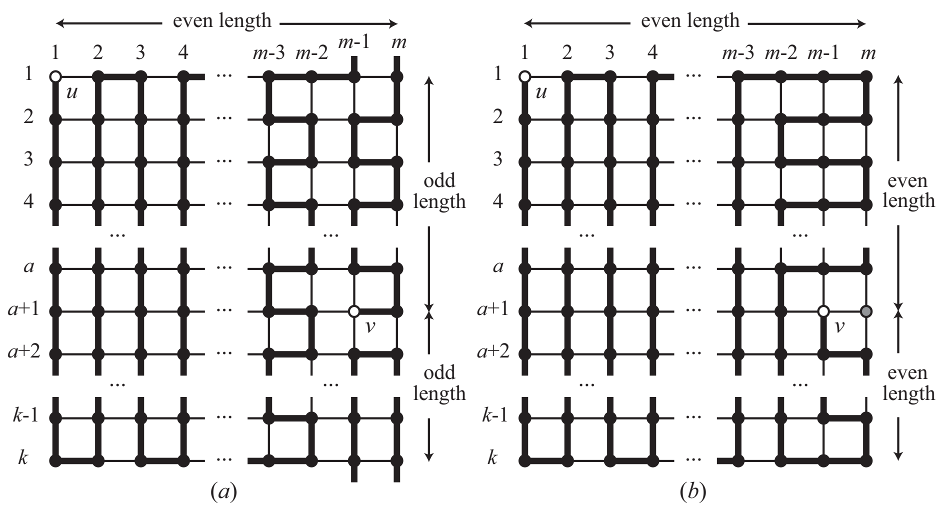

Figure 6. The different cases when $d(u^a, v)$ is odd.

Sub-case $(ii.b)$ Suppose that a is even (and so v lies on odd row $a + 1 \geq 1$ and on column $m - 1$). Consider the path ρ from u to v defined as:

$$(1,1),(2,1),\ldots,(k,1),(k,2),(k-1,2),\ldots,(1,2),$$

$(1,3),(2,3),\ldots,(k,3),(k,4),(k-1,4),\ldots,(1,4),$
$\ldots,(k,m-3),(k-1,m-3),\ldots,(1,m-3),(1,m-2),$
$(1,m-1),(1,m),(2,m),(2,m-1),(2,m-2),(3,m-2),(3,m-1),$
$(3,m),(4,m),(4,m-1),(4,m-2),\ldots,(a,m),(a,m-1),$
$(a,m-2),(a+1,m-2),(a+2,m-2),\ldots,(k,m-2),$
$(k,m-1),(k,m),(k-1,m),(k-1,m-1),(k-2,m-1),$
$\ldots,(a+2,m),(a+2,m-1),(a+1,m-1)$

(note that the vertex $(a+1,m)$ does not appear on ρ). The path ρ is almost-Hamiltonian and can be visualized as in Fig. 6(b). Similarly to as in the proof of Theorem 2, ρ can be progressively shortened to obtain paths of all odd lengths between $d(u,v)$ and k^n-2.

However, the base case is handled by Proposition 1. □

The following is an immediate corollary of Theorem 4.

Corollary 5 *Let $k \geq 3$ and $n \geq 2$, with k odd. Q_n^k is edge-bipancyclic.*

As remarked earlier, bipanconnectivity and bipancyclicity are concepts which make most sense in the context of bipartite graphs, such as the graphs Q_n^k, for k even. However, when k is odd, Q_n^k is not bipartite and it is possible that odd cycles might exist, as well as odd and even length paths between vertices u and v. As we shall see, this is indeed the case but not universally.

Henceforth, k is odd. Consider the vertices $u=(0,0,\ldots,0)$ and $v=(v_{n-1},v_{n-2},\ldots,v_0)$ of Q_n^k, where (as usual) we assume w.l.o.g. that $v_i \leq \frac{k-1}{2}$, for $i=0,1,\ldots,n-1$. Consider any path from u to v that does not use any 'wrap-around' edge, *i.e.*, an edge where the ith component of one incident vertex is $k-1$ and where the ith component of the other incident vertex is 0, for some i. Such a path must alternate between odd parity and even parity vertices; thus, such paths are either all of even length or all of odd length (depending upon whether $d(u,v)$ is even or odd). Suppose that $d(u,v)$ is odd (and so all such paths are of odd length). Let i be such that v_i is maximal from amongst $\{v_{n-1},v_{n-2},\ldots,v_0\}$. Any path from u to v of length at most

$$v_{n-1}+\ldots+v_{i+1}+(k-v_i-1)+v_{i-1}+\ldots+v_0=d(u,v)+k-2v_i-1$$

cannot use a wrap-around edge and so must be of odd length. Consequently, there are no even length paths from u to v of length less than $d(u,v)+k-2v_i$.

Identical reasoning implies that if $d(u, v)$ is even then there are no odd length paths from u to v of length less than $d(u, v) + k - 2v_i$. Consequently, we have a lower bound on the length of a shortest path, joining u and v and of parity different from that of $d(u, v)$.

Choose the vertex v of Q_n^k to be such that $v_{n-1} = 1$ and $v_j = 0$, for $j = 0, 1, \ldots, n-2$. Thus, there exists a vertex v such that $d(u, v)$ is odd and there are no paths joining u and v of even length less than $d(u, v) + k - 2$. There clearly also exists a vertex v' such that $d(u, v')$ is even and there are no paths joining u and v' of odd length less than $d(u, v) + k - 2$. Consequently, as we are interested in general statements concerning all pairs of distinct vertices from Q_n^k, we shall only look for even (resp. odd) length paths joining u and v of length at least $d(u, v) + k - 2$, when $d(u, v)$ is odd (resp. even).

Theorem 6 *Let $k \geq 3$ and $n \geq 2$, with k odd, and let u and v be distinct vertices of Q_n^k. There are paths joining u and v of all lengths in $\{i : d(u, v) + k - 3 \leq i \leq k^n - 1\}$. Furthermore, this result is optimal in that there exist distinct vertices u and v of Q_n^k for which $d(u, v)$ is odd (resp. even) and there are no even-length (resp. odd-length) paths joining u and v of length less than $d(u, v) + k - 2$.*

Proof The proof is very similar in structure to that of Theorem 4 and we adopt the exact same notation as in that proof (and in the proof of Theorem 2). There are two cases, according to whether $d(u^a, v)$ is odd or even. Given the earlier proofs, we are much briefer with our arguments here.

Case (i) $d(u^a, v)$ is even.

By Theorem 4, there exists a Hamiltonian path ρ_a from u^a to v in $Q_n^k(a)$ which can be progressively shortened to obtain paths of all even lengths between $d(u^a, v) = d(u, v) - a$ and $k^{n-1} - 1$, inclusive. As in the proofs of Theorems 2 and 4, the paths $\rho_0, \rho_1, \ldots, \rho_{k-1}$ yield a $k \times k^{n-1}$ grid, with rows $1, 2, \ldots, k$ and columns $1, 2, \ldots, m$, where $m = k^{n-1}$, with 'wrap-around' column edges.

Sub-case $(i.a)$ Suppose that a is even (and so v lies on odd row $a+1 \geq 1$ and on column m). Build the path ρ as depicted in Fig. 7(a). It is easy to see that ρ has length $k^n - 2$ and can be progressively shortened to obtain paths of all odd lengths between $(k-1) + d(u^a, v) + a + 1 = d(u, v) + k$ and $k^n - 2$ (shorten so that the resulting sub-path of length $k^{n-1} - 1$ lies on row k).

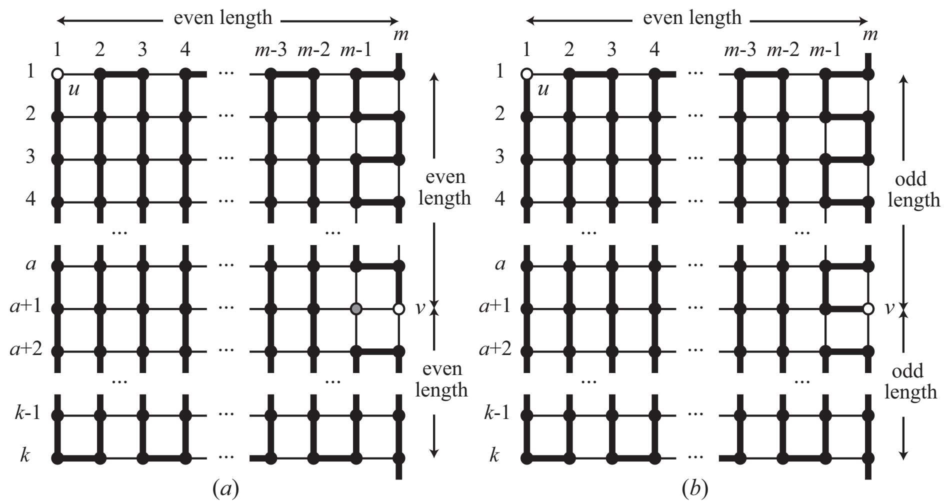

Figure 7. The different cases when $d(u^a, v)$ is even.

Sub-case $(i.b)$ Suppose that a is odd (and so v lies on even row $a+1 \geq 2$ and on column m). Build the path ρ as depicted in Fig. 7(b). It is easy to see that ρ has length $k^n - 1$ and can be progressively shortened to obtain paths of all even lengths between $(k-1)+d(u^a, v)+a+1 = d(u, v)+k$ and $k^n - 1$.

Case (ii) $d(u^a, v)$ is odd.

By Theorem 4, there exists an almost-Hamiltonian path ρ_a from u^a to v in $Q_n^k(a)$ which can be progressively shortened to obtain paths of all odd lengths between $d(u^a, v) = d(u, v)-a$ and $k^{n-1}-2$, inclusive, and so that the residual vertex is adjacent to v. As before, the paths $\rho_0, \rho_1, \ldots, \rho_{k-1}$ and the residual vertices yield a $k \times k^{n-1}$ grid, with rows $1, 2, \ldots, k$ and columns $1, 2, \ldots, m$, where $m = k^{n-1}$, with 'wrap-around' column edges.

Sub-case $(ii.a)$ Suppose that a is odd (and so v lies on even row $a+1 \geq 2$ and on column $m-1$). Build the path ρ as depicted in Fig. 8(a). It is easy to see that ρ has length $k^n - 2$ and can be progressively shortened to obtain paths of all odd lengths between $(k-1)+d(u^a, v)+a+1 = d(u, v)+k$ and $k^n - 2$.

Sub-case $(ii.b)$ Suppose that a is even (and so v lies on odd row $a+1 \geq 1$ and on column $m-1$). Build the path ρ as depicted in Fig. 8(b). It is easy to see that ρ has length $k^n - 1$ and can be progressively shortened to obtain paths of all even lengths between $(k-1)+d(u^a, v)+a+1 = d(u, v)+k$ and $k^n - 1$.

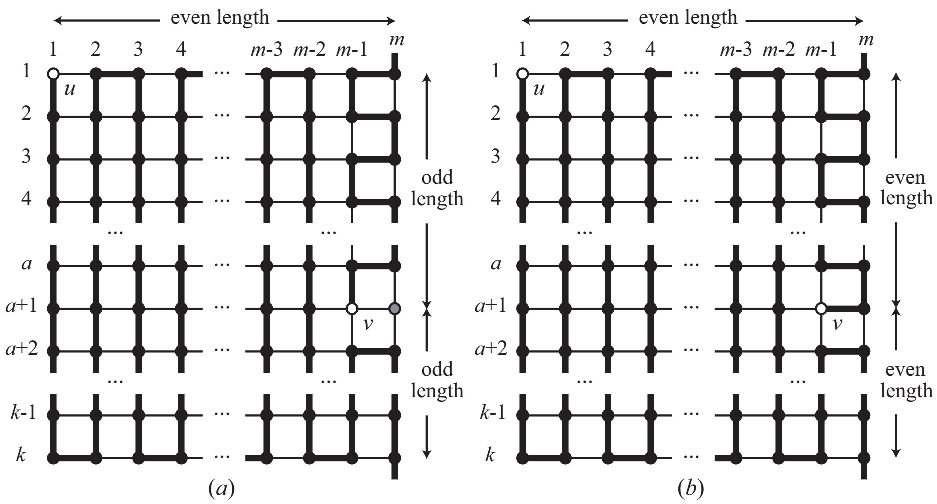

Figure 8. The different cases when $d(u^a, v)$ is odd.

In order to complete the construction of our paths, we deal with some special cases. W.l.o.g., assume that $v_{n-1} \neq 0$. There is trivially a path of length

$$(k - v_{n-1}) + v_{n-2} + \ldots + v_0 = d(u, v) + k - 2v_{n-1} \leq d(u, v) + k - 2$$

joining u and v. We can easily lengthen this path to obtain a path of length $d(u, v) + k - 2$ joining any distinct vertices u and v. Hence, no matter which vertex v is, Theorem 4 yields paths as in the statement of the result. Optimality follows by the argument presented prior to the statement of the result. □

Note that putting $k = 3$ in Theorem 6 yields the result from [15] that Q_n^3 is edge-pancyclic, and also resolves the question for arbitrary k, as was posed in [15]. The following corollary is immediate, given the fact that the diameter of Q_n^k, when k is odd, is $\frac{n(k-1)}{2}$.

Corollary 7 *Let $k \geq 3$ and $n \geq 2$, with k odd. The k-ary n-cube Q_n^k is m-panconnected, for $m = \frac{n(k-1)+2k-6}{2}$, and $(k-1)$-pancyclic.*

As remarked earlier, the bounds in Corollary 7 are optimal.

6 An application

We give here the outline of an application where we require our paths to be progressively shortened and where alternative shortening methods will not suffice.

Consider a parallel machine whose underlying interconnection network is a k-ary n-cube, and where this machine is required to solve problems specifically designed for a cycle of processors (amongst other problems), with the number of processors involved in the cycle being variable. Moreover, there is known to be a faulty processor in the machine and this faulty processor cannot be used in any embedded cycle. Furthermore, the location of the fault is not known and any cycle must be constructed in a distributed fashion, through message-passing between processors.

For simplicity, suppose that k is even and $n = 2$; consequently, any cycle we construct must have even length. We begin our construction by processor $(0, 0)$ attempting to construct a Hamiltonian path to processor $(0, 1)$ according to the construction in Proposition 1. Actually, the path is constructed as in Case 1.3 of Theorem 1 of [27]. It is important to note that the constructions in Proposition 1 (and Theorems 1 and 3 of [27]) are of such a uniform nature that the processor at the head of the path constructed so far can calculate in constant time the name of the next processor on the path, and can send a message to this processor thus extending the path constructed so far. If there were no faults then this construction would terminate with a Hamiltonian path from $(0, 0)$ to $(0, 1)$ laid out in the k-ary 2-cube. However, the construction will halt when the faulty processor is encountered (we assume that the processor immediately before the fault on the constructed path can detect that the next processor is faulty).

Let p be the processor that detects that the faulty processor is the next processor on the path, and suppose that this faulty processor is $f = (i, j)$. The processor p sends a message to processor $s = (i + 1, j)$ (over at most 4 hops, with addition modulo k) that it should use the construction of Proposition 1 to embark on the construction of a path of length k^2-2 to the processor $(i, j - 1)$. Note that the path, as shown in Fig. $2(b)$ (that is, the amended construction of a case from [27]), avoids the faulty processor f. We reiterate that the uniform nature of the construction is such that the processor at the head of the path constructed so far can calculate in constant time the name of the next processor on the path, and can send a message to this processor thus extending the path constructed so far. Having reached the processor

$(i, j-1)$, we actually truncate the path at processor $t = (i+1, j-1)$. Thus, we have a path of length $k^2 - 3$ from processor s to t, avoiding processor $(i, j-1)$ and the faulty processor f. Moreover, this path can be progressively shortened so as to obtain any odd length path (of length at most k^2-3) joining s to t (and avoiding f). Furthermore, again because of the uniformity of the construction and also the uniformity of the progressive shortening, this progressive shortening can easily be completed by message-passing between the processors. In fact, message-passing can be used so that every processor q on the path computes a list of triples of the form (q^+, q^-, i) detailing that q appears on a path of length i from s to t so that that the processor q^- (resp. q^+) is the next processor on this path moving towards s (resp. t). The existence of the edge (s, t) gives our embedded fault-avoiding cycles of varying lengths.

The above construction can be generalized to an analogous construction of fault-avoiding paths and cycles in Q_n^k where there is a faulty processor. As we stated above, we have not presented the precise details of this generalization; what suffices is that the general principle has been presented and any interested reader could implement the construction if needs be. We envisage that there are many other applications of progressive shortening but we have chosen not to explore these applications here.

7 Conclusions

In tandem with [15, 27], we have resolved completely the main questions concerning panconnectivity, bipanconnectivity, pancyclicity and bipancyclicity for a k-ary n-cube Q_n^k, when $k \geq 3$ and $n \geq 2$. In doing so, we have introduced the new concept of the progressive shortening of a path and shown how this concept can be used to solve a problem related to the embedding of linear arrays and cycles of processors in a distributed-memory multiprocessor whose interconnection network is a k-ary n-cube and where there is one faulty processor.

As directions for future research, we would like to see more applications of progressive shortening (and feel that the concept will prove to be more widely applicable). Also, we would like to see results on panconnectivity, pancyclicity, and so forth, extended to k-ary n-cubes in which there may be (a limited number of) faulty vertices or edges.

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