Node-to-Node Disjoint Paths in k-ary n-cubes with Faulty Edges

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***Abstract*—Let u and v be any two given nodes in a k-ary n-cube Q_n^k with at most $2n-2$ faulty edges. Suppose that the number of healthy links incident with u is no more than that of v, and denote this number by m. In this paper, we show that there are m mutually node-disjoint paths between u and v.**

***Keywords*-interconnection networks; disjoint paths; k-ary n-cube; fault tolerance.**

I. Introduction

The graph Q_n^k, known as the k-ary n-cube, has long been established as an interconnection network for distributed-memory parallel computers [1] and is an increasingly popular choice for networks-on-chips [2]. The k-ary n-cube has the following useful basic properties (in the context of parallel computing): it is vertex- and edge-symmetric [3]; it is Hamiltonian [4]; it has diameter $n\lfloor \frac{k}{2} \rfloor$; it has connectivity $2n$ [5]; and it has a recursive decomposition. Moreover, it has admirable properties in relation to routing, broadcasting and communication in general (see, for example, [3], [5]).

The one-to-one node-disjoint paths problem, where the aim is to find node-disjoint paths joining two given nodes in an interconnection network, is fundamental in parallel computing: such paths enable parallel communication between source and destination nodes; and provide alternative routing paths when faults occurs or for the purpose of avoiding heavy network traffic. Whilst Menger's Theorem [6] implies that there exist c node-disjoint paths between any two given nodes in a graph of node-connectivity c, it is not always easy to identify and actually construct the paths. Moreover, when there exist faults in some interconnection network, the problem of finding as many node-disjoint paths between any two given nodes becomes much more difficult.

Of course, as more and more processors are incorporated into parallel machines, faults become more common, be it faults in the processors or on the connections between processors. Given the significant cost of parallel machines, we would prefer to be able to tolerate (small numbers of) faults and still be able to use the parallel machine.

Bose et al. [5] construct $2n$ mutually node-disjoint paths between any two given nodes in a healthy k-ary n-cube. In this paper, we find mutually node-disjoint paths between any two given nodes in a faulty k-ary n-cube. In particular, let u and v be any two given nodes in a k-ary n-cube Q_n^k in which there are at most $2n-2$ faulty edges. Denote the number of healthy edges incident with any node u as $deg^h(u)$. We prove the existence of $\min\{deg^h(u), deg^h(v)\}$ mutually node-disjoint paths joining u and v.

The paper is organized as follows: the next section contains the basic definitions; we describe how to find node-disjoint paths in faulty k-ary 2-cubes in Section III; we use induction to find node-disjoint paths for k-ary n-cubes where $n \geq 3$ and $k \geq 4$ in Section IV; and finally, in Section V, we present our conclusions.

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II. Preliminaries

The k-ary n-cube, denoted by Q_n^k, for $k \geq 3$ and $n \geq 2$, has k^n nodes, indexed by the set $\{0, 1, \ldots, k-1\}^n$, and there is an edge $((u_n, u_{n-1}, \ldots, u_1), (v_n, v_{n-1}, \ldots, v_1))$ if, and only if, there exists $d \in \{1, 2, \ldots, n\}$ such that $min\{|u_d - v_d|, k - |u_d - v_d|\} = 1$, and $u_i = v_i$, for every $i \in \{1, 2, \ldots, n\} \setminus \{d\}$. Throughout, we assume that addition on tuple elements is modulo k.

An index $d \in \{1, 2, \ldots, n\}$ is often referred to as a *dimension*. We can *partition* Q_n^k *over dimension* d by fixing the dth element of any node tuple at some value x, for every $x \in \{0, 1, \ldots, k-1\}$. This results in k copies $Q_0, Q_1, \ldots, Q_{k-1}$ of Q_{n-1}^k (with Q_x obtained by fixing the dth element at x), with corresponding nodes in $Q_0, Q_1, \ldots, Q_{k-1}$ joined in a cycle of length k (in dimension d; we suppress d, n and k in the notation as they are always understood).

Let F be a set of faulty edges in Q_n^k, and let F_i the subset of faulty edges in the subgraph Q_i, for $i \in \{0, 1, \ldots, k-1\}$. F^v (resp. H^v) denotes the set of faulty (resp. healthy) edges that are incident with some node v, and F_i^v (resp. H_i^v) denotes the set of faulty (resp. healthy) edges that lie in Q_i and are incident with v. The set of edges incident with some node v is denoted by N^v. Let $v = (v_n, v_{n-1}, \ldots, v_2, v_1)$ be a node; so, it lies in Q_{v_1}. Denote the node $(v_n, v_{n-1}, \ldots, v_2, v_1 + i)$ as v^i, $-(k-1) \leq i \leq k-1$. In addition, v^1 (resp. v^{-1}) is also denoted as v^+ (resp. v^-). The distance between two node u and v is denoted by $dist(u, v)$. A path can be written as a list of nodes, as a list of nodes and edges, or as a list of sub-paths (of nodes and edges). We sometimes use square brackets to denote a path.

III. Node-to-Node Disjoint Paths in Faulty k-ary 2-cube, $k \geq 4$

We first find mutually node-disjoint paths in Q_2^4 when there are at most two faulty edges. Based on this, we find mutually node-disjoint paths in Q_2^k, where $k \geq 5$, where there are at most two faulty edges by mapping the graph Q_2^k to Q_2^4.

As the procedure of finding node-disjoint paths for Q_2^4 with two faulty edges is long and tedious, and due to space limitations, we give the following lemma with the proof omitted.

Lemma 1: In a 4-ary 2-cube Q_2^4 with at most two faulty edges, for any two given nodes u and v, there exist $\min\{deg^H(u), deg^H(v)\}$ mutually node-disjoint paths joining u and v.

In Q_2^k, a node associates with one row and one column, and an edge associates with one row and two columns, or one column and two rows. Denote the two faulty edges as f_1 and f_2. If there are at most four rows and at most four columns associated with u, v, f_1, and f_2, then we can reduce the situation to finding mutually node-disjoint paths in Q_2^4 by removing some of the non-associated rows and columns, and then extending the paths found by inserting back the rows and columns. So, we will only consider the cases that there are more than four rows or columns associated. We will consider the following cases:

(1) six columns are associated;
(2) five columns are associated.

The cases with more than four rows associated can be mapped to the above two cases by the symmetric properties of Q_n^k.

If there are 6 columns associated then there are at most 4 rows associated, and the configuration can be described as follows: $u = (0, 0)$, $v = (i_1, j_1)$, $f_1 = ((i_2, j_2), (i_2, j_3))$, $f_2 = ((i_3, j_4), (i_3, j_5))$, where $1 \leq i_1, i_2, i_3, j_1, j_2, j_3, j_4, j_5 \leq k, j_3 = j_2 + 1, j_5 = j_4 + 1$ and all the $j_t, 1 \leq t \leq 5$ are different from each other. To reduce the case to Q_2^4, we keep all of the associated rows and some other rows if necessary so as to obtain 4 rows; to have 4 columns, we set $j_1 = j_2 = 1, j_4 = 2$, if $j_1 < j_2 < j_4$, or we set $j_2 = 0, j_1 = j_4 = 2$, if $j_2 < j_1 < j_4$. The other cases can be easily mapped to these two cases by the symmetric properties of Q_2^k.

If there are 5 columns associated, the configuration can be described as: $u = (0, 0), v = (i_1, j_1), f_1 = ((i_2, j_2), (i_2, j_3)), f_2 = ((i_3, j_4), (i_4, j_4))$, where $1 \leq i_t, j_t \leq k, i_s \neq i_t, j_s \neq j_t$ for all $1 \leq s, t \leq 4$. To reduce the case to Q_2^4 (to have 4 columns), we set

$j_1 = 1, j_2 = 2, j_4 = 3$, if $j_1 < j_2 < j_4$, and set $j_2 = 0, j_1 = 2, j_4 = 3$, if $j_2 < j_1 < j_4$; by the symmetric properties of Q_2^k, the four rows can be kept in the same way as the columns if there are five rows associated, or we just keep all of the associated rows and some other rows if necessary so as to result in four rows. The other cases can be easily mapped to these two cases.

Thus, we have the following lemma.

Lemma 2: In a k-ary 2-cube Q_2^k with at most two faulty edges, for any two given nodes u and v, there exist $\min\{deg^H(u), deg^H(v)\}$ mutually node-disjoint paths joining u and v.

IV. Node-to-Node Disjoint Paths in Faulty k-ary n-cubes, $n \geq 4, k \geq 4$

Suppose that there are at most $2n - 2$ faulty edges in Q_n^k. Hence, we can partition Q_n^k over some dimension so that there is at most one faulty edge lying in that dimension.

Given an edge $e = (s, t)$ on some path, define *U-jump(e)* as replacing the edge e by a sub-path of length 3: $[s, s^-, t^-, t]$; similarly, *D-jump(e)* is defined as replacing e by another sub-path of length 3: $[s, s^+, t^+, t]$.

W.l.o.g., suppose that we partition Q_n^k over dimension one so that there is at most one faulty edgein this dimension.

There are a number of cases to be considered, and the proofs are technical and complicated. For example, supposing that w.l.o.g. $u = (0, 0, \ldots, 0)$, we need to consider several different cases depending on where the node $v = (v_n, v_{n-1}, \ldots, v_1)$ lies and whether there is a faulty edge in dimension one or not. Suppose that there is one faulty edge lying in dimension one. We have to distinguish the value of v_1:

(1) $v_1 = 0$, which means u and v lie in the same sub-graph Q_0;

(2) $v_1 \neq 0$, which means u and v lie in different sub-graphs.

The techniques in building node-disjoint paths for these two cases are very different. For example, for case (1), we can use induction on Q_0 by assuming some of the faulty edges are healthy if necessary so as to have at most $2n - 4$ faulty edges (if a path includes an edge, say e, that is originally faulty then we need to do some adjustments to avoid the faulty edge e; for example, do a *U-jump(e)* or *D-jump(e)*), and then find some more paths to 'link' the dimension one edges, while for case (2), as u and v lie in different sub-graphs, we can apply induction on each sub-graph by making some assumptions (however, we need to link these disjoint paths, in different sub-graphs, together which makes the procedure complicated).

Furthermore, if we go into details for case (2), we have the following sub-cases to consider:

(2.1) $dist(u, v^{-v_1}) = 0$, which means the node u and the node v lie in the same dimension one circle. Node u or its neighbor can reach node v or its corresponding neighbor along the dimension one circle.

(2.2) $dist(u, v^{-v_1}) = 1$, which means node u and node v lie on different dimension one circles, but they have one overlap incident edge (u, v^{-v_1}).

(2.3) $dist(u, v^{-v_1}) \geq 2$, which means node u and node v lie on different dimension one circles, and they have no overlap incident edges.

These cases are similar to each other; however, there is still some variation. Due to space limitations, we will not include all of the proof in the paper, only the representative parts, and the remainder will be available in an expanded version of this paper.

We claim that case (2.2) is representative as most of the techniques used in the other cases will be used here. For example, we use induction on Q_0 to find disjoint paths, which is similar to case (1); we extend these disjoint paths to Q_{v_1}, which is similar to case (2.1) when $\exists e_1 \in F^u, e_1^{v_1} \notin F^v$ and $\exists e_2 \notin F^u, e_2^{v_1} \in F^v$, and case (2.3). Also, similar discussion on the case when the dimension one faulty edge does not block any path can be applied to the case when there are no dimension one faulty edges.

In what follows, we will discuss case (2.2) in detail.

Case (2.2). The conditions are:

(1) There is one dimension one faulty edge $f = (r^h, r^{h+1})$;
(2) $dist(u, v^{-v_1}) = 1$, i.e., there exists some $j \in \{2, 3, \ldots, n\}, v_j = 1$, and for all $i \in \{2, 3, \ldots, n\} \setminus \{j\}, v_i = 0$. W.l.o.g., suppose $j = 2$.

Let $e \in N^t$. For a given $i \in \{1, 2, \ldots, k-1\}$, we define $map(t, i)$ as temporarily marking e^i as faulty if e is faulty, or healthy otherwise.

Suppose r, s lie in the same sub-graph $Q_i, i \in \{0, 1, \ldots, k-1\}$, and $|F_i| \leq 2n-4$. Denote $DP(r, s)$ as a procedure of finding node-disjoint paths joining r and s in the sub-graph Q_i by induction.

Note that if the following two paths exist, and are not blocked, we will always keep them in the set of disjoint paths, and thus we may not mention them in the following discussion unless necessary:

$\sigma_1 : [u, v^-, v]$; and
$\sigma_2 : [u, u^+, v]$.

(A22-1) Assume f is healthy.

As there are now at most $2n-3$ faulty edges in Q_n^k, we have the following cases to consider. Recall that we suppose $|F^u| \geq |F^v|$.

Suppose $v_1 = 1$.

(A22-1.1) $|F^u| = 2n-3$.

We only need to find 3 disjoint paths. Let (u, s) be the only healthy edge in Q_0 that is incident with u. The two extra paths are:

(a) $[u, u^-, v^{-2}, v^-, v]$; and
(b) $[u, s, s^-, \ldots, s^2, u^2, v^2, v^+, v]$;

(A22-1.2) After applying $map(v, 0)$, we have $|F_0| = 2n-3$.

If $|F^u \cup F^{v^-}| = 2n-3$, we must have $|F^u| \geq |F^{v^-}| + 1$. Assume a faulty edge (u, s) is healthy, and apply $DP(u, v^-)$. Remove the node v^- from each disjoint path, and extend the path to v. For simplicity, this procedure after applying $DP(u, v^-)$ will be called "extending" the paths to v throughout the remainder part of this section. Two more paths can be defined as:

(a) $[u, u^+, u^2, v^+, v]$; and
(b) $[u, u^-, v^{-2}, v^-, v]$.

If there is a faulty edge e that is neither incident with u nor with v^{-v_1}, assume e is healthy and apply $DP(u, v^-)$; extend the paths to v. Apply ***U-jump(e)*** if e lies on some of the disjoint paths. Two additional paths can be defined as:

(a) $[u, u^+, u^2, v^+, v]$; and
(b) $[u, u^-, v^{-2}, v^-, v]$;

(A22-1.3) $|F_1| \leq 2n-4$, and after applying $map(v, 0)$, we have $|F_0| \leq 2n-4$.

Apply $map(v, 0)$ and $DP(u, v^-)$. Extend the paths to v. Two additional paths can be defined as:

(a) $[u, u^+, u^2, v^+, v]$; and
(b) $[u, u^-, v^{-2}, v^-, v]$;

From our above construction method, in a similar way to the case $v_1 = 1$ but extending the dimension 1 edges, we will be able to find disjoint paths for the case $v_1 \geq 2$.

If the dimension 1 edge f lies on some of the above paths, we will rebuild disjoint paths as follows.

(A22-2) f is blocking some path.

Suppose $v_1 = 1$.

(A22-2.1) $f = (u, u^+)$.

(A22-2.1.1) $|F^u| = |F^v|$

We first apply $map(v, 0)$. But (u, v^-) is marked as faulty if either (u, v^-) is originally faulty or (u^+, v) is faulty.

If $(u, v^-) \in F$, and $(u^+, v) \notin F$, assume two of v^-'s faulty links e_1, e_2 are healthy. Apply $DP(u, v^-)$. Let the edge (u, w) lie on the path that e_1 lies on, and (u, z) lie on the path that e_2 lies on. Remove the paths that contain e_1 or e_2. For all of the other paths, extend them to v. Build the following three paths:

(a) $[u, u^-, \rho_1, v^{-2}, v^-, v]$;
(b) $[u, z, z^+, z^2, \rho_2, v^+, v]$; and
(c) $[u, w, w^+, \rho_3, u^+, v]$,

where $\rho_1 \in Q_{k-1}, \rho_2 \in Q_2$ and $\rho_3 \in Q_1$. These paths can be found by induction. Specifically, ρ_3 can be built by finding disjoint paths between w^+ and v (note that there are at most $2n-4$ faulty edges lying in Q_1).

If $(u, v^-) \notin F$, and $(u^+, v) \in F$, apply $map(v, 0)$. If there are more than $2n-4$ faulty edges in Q_0, we assume that (u, v^-) is healthy.

Apply $DP(u, v^-)$; remove path $[u, v^-]$ and extend the paths to v. Build one more path:

$$[u, u^-, \ldots, u^{v_1+1}, \rho, v^+, v],$$

where $\rho \in Q_{v_1+1}$.

If either $(u, v^-) \in F$ and $(u^+, v) \in F$, or $(u, v^-) \notin F$ and $(u^+, v) \notin F$, apply $map(v, 0)$; assume $e \in F^{v^-}$ is healthy. Apply $DP(u, v^-)$. Let (u, w) be on the path containing e. Remove this path, and extend all other paths to v. Build the following two paths:

(a) $[u, w, w^+, \rho_1, u^+, v]$, where ρ can be chosen from the disjoint paths between w^+ and v in Q_1; and

(b) $[u, u^{k-1}, \ldots, u^{v_1+1}, \rho_2, v^+, v]$, where $\rho_2 \in Q_{v_1+1}$ and can be found by induction.

If $(u, v^-) \in F$ and $(u^+, v) \in F$ then path (a) is not needed.

(A22-2.1.2) $|F^u| > |F^v|$

(A22-2.1.2.1) $(u, v^-) \in F$ and $(u^+, v) \notin F$.

If $|F^u| = |F^v| + 1$, then after applying $map(v, 0)$, we have $|F_0^u| = |F_0^{v^-}| + 1$. Assume the faulty edge e that is incident with v^- is healthy. Apply $DP(u, v^-)$. Let (u, w) lie on the path that passes through the edge e. Remove this path and extend all other paths to v. Build the following two paths:

(a) $[u, w, w^-, \rho_1, v^{-2}, v^-, v]$, where $\rho_1 \in Q_{k-1}$ can be found by finding disjoint paths between u^- and v^{-2} by induction, and

(b) $[u, u^-, \ldots, u^{v_1+1}, \rho_2, v^+, v]$, where $\rho_2 \in Q_{v_1+1}$

If $|F^u| \geq |F^v| + 2$, then after applying $map(v, 0)$, assume one of the faulty edges e that is incident with u is healthy. Apply $DP(u, v^-)$. Remove the path that passes through the edge e and extend all other paths to v. Build the following path:

$$[u, u^-, \ldots, u^{v_1+1}, \rho_2, v^+, v],$$

where $\rho_2 \in Q_{v_1+1}$.

(A22-2.1.2.2) $(u, v^-) \notin F$ and $(u^+, v) \in F$; or $(u, v^-) \in F$ and $(u^+, v) \in F$; or $(u, v^-) \notin F$ and $(u^+, v) \notin F$.

Apply $map(v, 0)$. If there are at most $2n - 4$ faulty edges in Q_0, apply $DP(u, v^-)$, and extend the paths to v. Build another path:

$$[u, u^-, \ldots, u^{v_1+1}, \rho_2, v^+, v],$$

where $\rho_2 \in Q_{v_1+1}$.

If there are $2n - 3$ faulty edges in Q_0 and $(u, v^-) \in F$ after applying $map(v, 0)$, we assume that (u, v^-) is healthy and apply $DP(u, v^-)$. Remove the path $[u, v^-]$. Extend all other paths to v, and build one more path as above.

If there are $2n - 3$ faulty edges in Q_0 and $(u, v^-) \notin F$ after applying $map(v, 0)$, we have $(u, v^-) \notin F$ and $(u^+, v) \notin F$. Hence, we assume one of v^-'s faulty edges e is healthy and apply $DP(u, v^-)$. Remove the path that passes though the edge e and extend all other paths to v. Build one more path as above.

(A22-2.2) $f = (u, u^-)$.

(A22-2.2.1) $|F^u| = |F^v|$

(A22-2.2.1.1) $(u, v^-) \in F$ and $(u^+, v) \notin F$.

Apply $map(v, 0)$, and assume the faulty edges e_1 and e_2 incident with v^- are healthy. Apply $DP(u, v^-)$. Let (u, w) lie on the path that passes through e_1, and let (u, z) lie on the path that passes through e_2. Remove these two paths and extend all other paths to v. Build the following two paths:

(a) $[u, w, w^-, \rho_1, v^{-2}, v^-, v]$, where $\rho_2 \in Q_{k-1}$, and

(b) $[u, z, z^+, z^2, \rho_2, v^+, v]$, where $\rho_2 \in Q_2$.

ρ_1 and ρ_2 can be built by induction in their corresponding sub-graphs.

(A22-2.2.1.2) $(u, v^-) \notin F$ and $(u^+, v) \in F$

Apply $map(v, 0)$ but mark (u, v^-) as healthy, and apply $DP(u, v^-)$. Extend the paths to v. Build one more path:

$$[u, u^+, u^2, \rho_2, v^+, v],$$

where $\rho_2 \in Q_2$.

(A22-2.2.1.3) $(u, v^-) \in F$ and $(u^+, v) \in F$.

Apply $map(v, 0)$ and $DP(u, v^-)$. Extend the disjoint paths to v. Build the same extra path as in case (A22-2.2.1.2) .

(A22-2.2.1.4) $(u, v^-) \notin F$ and $(u^+, v) \notin F$.

Apply $map(v, 0)$. There exists one faulty edge e incident with v^-. We assume that e is healthy, and apply $DP(u, v^-)$. Remove the path that passes through the edge e (if such path exists), and extend all other paths to v.

(A22-2.2.2) $|F^u| > |F^v|$

(A22-2.2.2.1) $(u,v^-)\in F$ and $(u^+,v)\notin F$.

Apply $map(v,0)$. If $|F^u| = |F^v|+1$ then assume that edge e, a faulty edge incident with v^-, is healthy, and apply $DP(u,v^-)$. Let (u,w) lie on the path that passes through e. Remove this path and expand all other paths to reach v. Build one more path:

$$[u,w,w^-,\rho,v^{-2},v^-,v].$$

If $|F^u|\geq |F^v|+2$, assume that the edge (u,v^-) is healthy, and apply $DP(u,v^-)$. Remove the path $[u,v^-]$, and extend all other paths to v.

(A22-2.2.2.2) $(u,v^-)\notin F$ and $(u^+,v)\notin F$.

Apply $map(v,0)$. If there are no more than $2n-4$ faulty edges in Q_0 then apply $DP(u,v^-)$, and extend the disjoint paths to v.

If there are $2n-3$ faulty edges in Q_0 and $|F_0^u| = |F_0^{v^-}|$, then there must exist e that is neither incident with u nor incident with v^-. Assume e is healthy and apply $DP(u,v^-)$. If some path passes edge e, apply ***U-jump(e)***. Extend these paths to v.

(A22-2.2.2.3) $(u^+,v)\in F$.

No matter whether (u,v^-) is faulty or not, we apply $map(v,0)$, and assume that (u,v^-) is healthy and apply $DP(u,v^-)$. Extend the disjoint paths to v. Build one more path:

$$[u,u^+,u^2,\rho,v^+,v].$$

(A22-2.3) $f=(v,v^-)$.

If $(u,v^-)\notin F$ and $(u^+,v)\in F$ then we assume that two of u's incident faulty edges e_1 and e_2 (lying in Q_0) are healthy. Apply $DP(u,v^-)$. Suppose w_1 (resp. w_2) lies on the path that edge e_1 (resp. e_2) lies on, where w_1 and w_2 are neighbors of v^-. Remove these two paths, extend all other paths to v, and build the following three paths:

(a) $[u,v^-,v^{-2},\ldots,v^+,v]$;
(b) $[u,u^-,\rho_1,w_1^-,w_1,w_1^+,v]$, $\rho_1\in Q_{k-1}$; and
(c) $[u,u^+,u^2,\rho_2,w_2^2,w_2^+,v]$, $\rho_2\in Q_2$.

Based on the above paths, if $(u^+,v)\notin F$ then there is no need to build the above path (c), as u,u^+,v is already one of the disjoint paths; if $(u,v^-)\in F$ then there is no need to build the above path (a).

(A22-2.4) $f=(v,v^+)$.

(A22-2.4.1) $(u,v^-)\notin F$ and $(u^+,v)\in F$.

There exist faulty edges e_1 and e_2 incident with node u. Assume that they are healthy, and apply $DP(u,v^-)$. Let w_i lie on the path that passes thorugh edge e_i, for $i=1,2$. Remove the path that passes through e_1 and e_2. Extend all of the other paths to v. Build the following two additional paths:

(a) $[u,u^-,\rho_1,w_1^-,w_1,w_1^+,v]$, $\rho_1\in Q_{k-1}$;
(b) $[u,u^+,u^2,\rho_2,w_2^2,w_2^+,v]$, $\rho_2\in Q_2$.

(A22-2.4.2) $(u,v^-)\notin F$ and $(u^+,v)\notin F$.

Similar to case (A22-2.4.1), but only assume one faulty edge e_1 is healthy; we only need to find one more path (a) as in case (A22-2.4.1).

(A22-2.4.3) $(u,v^-)\in F$ and $(u^+,v)\in F$.

There exists $e_1\in F^u$. Assume e_1 is healthy, and apply $DP(u,v^-)$. Let w_1 lie on the path that passes through the edge e_1. Remove this path and extend all other paths to v. Build the following two paths:

(a) $[u,u^-,\rho_1,v^{-2},v_-,v]$;
(b) $[u,u^+,u^2,\rho_2,w_1^2,w_1^+,v]$.

(A22-2.4.4) $(u,v^-)\in F$ and $(u^+,v)\notin F$.

Similar to case (A22-2.4.3), but no need to make any assumption, except when $|F_0|=2n-3$ after applying $map(v,0)$. If the exceptional case happens then we assume that (u,v^-) is healthy and then apply $DP(u,v^-)$. Extend the paths to v.

(A22-2.5) $f=(u^h,u^{h+1})$, or $f=(v^h,v^{h+1}), h\neq 0,-1$.

In this case, the two paths below are included in the node-disjoint paths set:

(a) $[u,v^-,v]$; and
(b) $[u,u^+,v]$.

We need to find a path linking edge (u,u^-) and edge (v,v^+) but avoiding the faulty edge f. It can be either

(a) $[u,u^-,\rho_1,v^{-2},v_-,v]$, $\rho_1\in Q_{k-1}$,

if $f=(u^h,u^{h+1})$ for some $h\neq 0,-1$, or

(b) $[u,u^+,u^2,\rho_2,w_1^2,w_1^+,v]$, $\rho_2\in Q_2$,

if $f=(v^h,v^{h+1})$ for some $h\neq 0,-1$.

The other paths can be built similar to case (A22-1).

As we assume $v_1=1$ at the beginning of case (A22-2), we now consider $v_1>1$.

Compare to the case (A22-1), where we assume f is healthy and when $v_1 > 1$; here we only need to consider the case when $f = (r^h, r^{h+1})$ with $0 \leq h \leq v_1$ but f is neither incident with u nor incident with v, and blocks some of the paths as constructed in case (A22-1) (when $v_1 > 1$). We consider the following cases.

- If $r = u$ then there exists s, a neighbor of u, such that (u^j, s^j) is healthy for every $j = 0, 1, \ldots, k-1$. This is true, as there are at most $2n-3$ faulty edges in all $Q_i, i = 0, 1, \ldots, k-1$, and each node in Q_i has $2n-2$ neighbors inside Q_i. Thus, we adjust the blocked path by replacing the faulty edge with a sub-path: $[r^h, s^h, s^{h+1}, r^{h+1}]$. Note that if $h = v_1 - 1$ or $h = 0$, some slight adjustment is necessary, which will be simple and is not stated here.
  However, it might be the case that the only healthy edge in Q_h for node r^h is v^{h-v_1}, and that v^{h-v_1} is already on some path. In this case, we partition the graph Q_n^k over dimension 2. This will result in one dimension two faulty edge. For convenience, we also call a sub-graph Q_i if the dimension two number is i; we also denote the node $(v_n, v_{n-1}, \ldots, v_2 + i, v_1)$ as v^i if $v = (v_n, v_{n-1}, \ldots, v_2, v_1)$, for $-k+1 \leq i \leq k-1$. Now we have $dist(u, v^-) > 1$, and the only dimension 2 faulty edge is: $((0, 0, \ldots, 1, i), (0, 0, \ldots, 2, i))$. As there are no faulty edges in the sub-graph Q_0, we find disjoint paths by induction between u and $(0, 0, \ldots, 0, v_1)$, and extend these paths to v. Thus we have found $2n-4$ disjoint paths. Two more paths can be defined as:
  (a) $[u, u^-, \rho_1, v^{-2}, v^-, v]$, $\rho_1 \in Q_{k-1}$ can be found by induction;
  (b) $[u, u^+, u^2, \rho_2, v^+, v]$, $\rho_2 \in Q_2$ can be found by induction.

  The case that node r is a neighbor of v^- can be done similarly to the case $r = u$.
- If $r = v^{-v_1}$, and the only healthy edge incident with r^h is u^h, we then repartition the graph similarly to the case above. Otherwise, there exists $(r^h, s^h) \notin F$ and $(r^{h+1}, s^{h+1}) \notin F$. Replace the faulty edge f by a sub-path of length three: $[r^h, s^h, s^{h+1}, r^{h+1}]$. If (s^h, s^{h+1}) is already on some path and whose dimension one part is $s^x, s^{x+1}, \ldots, s^y$ then we replace these edges by $s^x, s^{x-1}, \ldots, s^{y+1}, s^y$.

Based on Lemma 1, Lemma 2 and the above discussion, we obtain the following result.

Theorem 1: In a k-ary n-cube with at most $2n-2$ faulty edges, there exist $\min\{deg^H(u), deg^H(v)\}$ mutually node-disjoint paths joining any two given nodes u and v.

V. Conclusion

We have proved that, for any two given nodes u and v in a k-ary n-cube, where there are at most $2n-2$ faulty edges, there exist $\min\{deg^H(u), deg^H(v)\}$ mutually node-disjoint paths between u and v. Thus, the k-ary n-cube interconnection network is robust as regards communication and routing, even when we have up to $2n-2$ faulty edges. Our future research will focus on finding shortest node-disjoint paths for k-ary n-cubes with faulty edges and/or faulty nodes.

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