The expressibility of fragments of Hybrid Graph Logic on finite digraphs

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Abstract

Hybrid Graph Logic is a logic designed for reasoning about graphs and is built from a basic modal logic, augmented with the use of nominals and a facility to verify the existence of paths in graphs. We study the finite model theory of Hybrid Graph Logic. In particular, we develop pebble games for Hybrid Graph Logic and use these games to exhibit strict infinite hierarchies involving fragments of Hybrid Graph Logic when the logic is used to define problems involving finite digraphs. These fragments are parameterized by the quantifier-rank of formulae along with the numbers of propositional symbols and nominals that are available. We ascertain exactly the relative definability of these parameterized fragments of the logic.

1 Introduction

Graphs of one form or another are ubiquitous in mathematics and computer science, and almost all logics can be used to reason about them. One particular application area of graph-based reasoning is in model checking where we are given a formal model of some system and a property of that system, and we wish to verify whether the given model has the given property. In model checking, the formal model (which may be infinite) is usually supplied as a labelled digraph in the form of a Kripke structure and the property is usually expressed by some formula of a modal or temporal logic (see, for example, [7]; throughout, by 'graph' we mean directed graph, which is a normal Kripke frame). Key to the use of logics for model checking is the decidability and complexity of the related model-checking, satisfiability and validity problems.

Hybrid logics go back to the work of Prior (see, for example, [1]) but have only relatively recently been studied in relation to computer science (see, for example, [1, 10, 15]). A hybrid logic is an extension of a modal or temporal logic in which symbols are used to name individual points in Kripke structures. Key to many hybrid logics is the use of nominals, n, and nominal operators, $@_n$, which allow us to 'jump' to the point of a Kripke structure named by the nominal n, and the 'binder' $\downarrow$, which allows us to bind variables to points. The study of hybrid logics in theoretical computer science has been in a number of contexts, such as in relation to description logics used in knowledge representation (for example, [5]), to proof theory (for example, [6]) and to model checking (for example, [10]) where the focus has been on the decidability and complexity of the related model-checking, satisfiability and validity problems.

Finite model theory is the study of the model theory of finite structures and, of course, has a strong relationship with fields such as model checking. One thriving aspect of finite model theory is the classification of logics according to their capacity to define problems (that is, isomorphism-closed classes of finite structures), particularly in relation to the computational complexity of the problems; this sub-area of finite model theory is sometimes referred to as descriptive complexity. Up until recently, hybrid logics have not been closely studied in this context; that is, as mechanisms for defining isomorphism-closed classes of (finite) frames (that is, digraphs). However, in [4] Benevides and Schechter defined (amongst other hybrid logics) *Hybrid Graph Logic HGL*, which is a very basic modal logic augmented with the use of nominals and a facility to verify the existence of paths in graphs through the (path-) quantifiers $\Diamond^+$ and $\Box^+$. The intention in [4] was to develop (modal and hybrid) logics for reasoning about graphs (that is, frames, as opposed to Kripke structures) that are expressive enough to define core graph-theoretic problems relating to properties such as connectedness, acyclicity and Hamiltonicity. It should be added that a transitive closure operator has also been added to hybrid logics in the form of Zhen and Seligman's 'community operator' [16] and recently by Lange too [14].

Actually, Hybrid Graph Logic *without* nominals is a well-known and well-studied fragment of both PDL and CTL (see, for example, [1, 9]); furthermore, the logic HGL itself (that is, where nominals are allowed) has been independently formulated and studied (from the perspective of tableaux systems) in [13] where it is referred to as basic modal logic extended with nominals and eventualities, and denoted H^*. We continue to refer to the logic as HGL, given that the emphasis of our study follows the tone set in [4].

In this paper, we focus on Hybrid Graph Logic HGL as a logic for defining problems involving finite directed graphs (that is, finite frames). We develop an Ehrenfeucht-Fraïssé-style pebble game for HGL that allows us to study the expressibility of fragments $\text{HGL}_r(c,d)$ of the logic HGL, parameterized by the quantifier-rank r of formulae, the number c of propositional symbols available and the number d of nominals available. We use our pebble game to show that for any $r, c, d \geq 0$, when we equate a logic with the class of (digraph) problems it defines, we have that

$$\text{HGL}_r(c,d) \text{ is a proper subset of } \text{HGL}_{r+1}(c,d);$$

in fact, in addition we show that there are problems definable by formulae of $\text{HGL}_{r+1}(c,d)$ in which the path-quantifiers $\Diamond^+$ and $\Box^+$ are not used. Moreover, we also show that if $r \geq 1$, $c, d \geq 0$, $c' \geq c$, $d' \geq d$ and $c' + d' = c + d + 1$ then

$$\text{HGL}_r(c,d) \text{ is a proper subset of } \text{HGL}_r(c',d')$$

(and we detail exactly the problems definable in the logics $\text{HGL}_0(c,d)$). Consequently, we obtain a refined view of the structure of Hybrid Graph Logic and ascertain the relative expressive power of the logics formed by restricting the quantifier-rank, the number of propositional symbols and the number of nominals.

In the next section, we give the basic definitions and notation relevant to this paper before developing our pebble games in Section 3. In Section 4, we play our games and obtain our hierarchy results. Our conclusions are given in Section 5.

2 Basic definitions

In this section, we recapitulate the syntax and semantics of Hybrid Graph Logic (as it was defined in [4]). In essence, Hybrid Graph Logic is a hybrid logic augmented with a facility to validate paths in structures, and the structures in which the formulae of Hybrid Graph Logic are interpreted are Kripke structures with a single modality. We explain how we use the semantics of Hybrid Graph Logic so as to work with finite digraphs through the consideration of Kripke structures. Whilst our presentation is self-contained, we refer the reader to [1] and [11] for more information as regards hybrid logics and modal logics, respectively.

2.1 The syntax and semantics of HGL

First, the syntax of Hybrid Graph Logic. Every formula of Hybrid Graph Logic is parameterized by a set of *propositional symbols* P (coming from some set of available propositional symbols) and by a set of *nominals* N (coming from some set of available nominals).

Definition 1 The formulae: p, where p is a propositional symbol; n, where n is a nominal; and $\perp$ are well-formed formulae of Hybrid Graph Logic and are atomic formulae. If ψ and ψ' are well-formed formulae of Hybrid Graph Logic then so are

$$\psi \Rightarrow \psi' \ ; \ \Diamond\psi \ ; \ \Diamond^+\psi \ ; \ \text{ and } @_n\psi,$$

where n is a nominal. The language $\text{HGL}(\mathsf{P}, \mathsf{N})$ consists of those formulae, built recursively as stated here, for which every propositional symbol used comes from the finite set P and every nominal used comes from the finite set N.

Now for the structures in which we interpret formulae of $\text{HGL}(\mathsf{P}, \mathsf{N})$.

Definition 2 We write $\mathcal{G} = (V, E)$ when $\mathcal{G}$ is a digraph with (non-empty finite) vertex set V and edge set E (we allow the possibility of self-loops). When we think of $\mathcal{G}$ as a relational structure $\langle V, E\rangle$, with E a binary relation, we refer to $\mathcal{G}$ as a *frame*, with the vertices of the frame $\mathcal{G}$ referred to as *points*. A *pointed frame* $\langle \mathcal{G}, v\rangle$ is a frame $\mathcal{G} = \langle V, E\rangle$ together with a point $v \in V$. Given a set of propositional symbols P and a set of nominals N, a *Kripke* $\mathsf{P} \cup \mathsf{N}$*-structure* $\mathcal{C} = \langle \mathcal{G}, \mu\rangle$ is a frame $\mathcal{G} = \langle V, E\rangle$ together with a function $\mu : \mathsf{P} \cup \mathsf{N} \to \wp(V)$, called a *valuation function*, for which for every $n \in \mathsf{N}$, $\mu(n)$ is a singleton set (if $\mu(n) = \{v\}$ then we sometimes write $\mu(n) = v$ and say that the nominal sits on the point v). The points of $\langle \mathcal{G}, \mu\rangle$ are simply the points of $\mathcal{G}$. A *pointed Kripke* $\mathsf{P} \cup \mathsf{N}$*-structure* $\langle \mathcal{C}, v\rangle$ is a Kripke $\mathsf{P} \cup \mathsf{N}$-structure $\mathcal{C} = \langle \mathcal{G}, \mu\rangle$ together with a point v of $\mathcal{G}$.

We use the terms 'digraph' and 'frame' and the terms 'vertex' and 'point' interchangeably, depending upon the context of our conversation.

Finally, we give semantics to the Hybrid Graph Logic HGL(P, N) (we refer to both the language and the logic by HGL(P, N)).

Definition 3 Let φ be a formula of Hybrid Graph Logic that involves propositional symbols from the set P and nominals from the set N. We interpret φ in a pointed Kripke $\mathsf{P} \cup \mathsf{N}$-structure $\langle\langle \mathcal{G} = \langle V, E\rangle, \mu\rangle, v\rangle$ as follows.

1. $\langle\langle \mathcal{G}, \mu\rangle, v\rangle \models p$ if, and only if, $v \in \mu(p)$.
2. $\langle\langle \mathcal{G}, \mu\rangle, v\rangle \models n$ if, and only if, $v = \mu(n)$.
3. It is not the case that $\langle\langle \mathcal{G}, \mu\rangle, v\rangle \models \bot$.
4. If φ is of the form $\psi \Rightarrow \psi'$ then $\langle\langle \mathcal{G}, \mu\rangle, v\rangle \models \varphi$ if, and only if, whenever $\langle\langle \mathcal{G}, \mu\rangle, v\rangle \models \psi$, we must have that $\langle\langle \mathcal{G}, \mu\rangle, v\rangle \models \psi'$.
5. If φ is of the form $\Diamond\psi$ then $\langle\langle \mathcal{G}, \mu\rangle, v\rangle \models \varphi$ if, and only if, there exists some point v' of $\mathcal{G}$ for which $E(v, v')$ holds in $\mathcal{G}$ and for which $\langle \mathcal{G}, v'\rangle \models \psi$.
6. If φ is of the form $\Diamond^+\psi$ then $\langle\langle \mathcal{G}, \mu\rangle, v\rangle \models \varphi$ if, and only if, there exist points $v_0, v_1, \ldots, v_q$, where $q \geq 1$, for which: $v = v_0$; $E(v_i, v_{i+1})$ holds in $\mathcal{G}$, for all $i = 0, 1, \ldots, q-1$; and $\langle \mathcal{G}, v_q\rangle \models \psi$.
7. If φ is of the form $@_n\psi$ then $\langle\langle \mathcal{G}, \mu\rangle, v\rangle \models \varphi$ if, and only if, $\langle\langle \mathcal{G}, \mu\rangle, u\rangle \models \psi$, where $u = \mu(n)$.

If $\langle\langle \mathcal{G}, \mu\rangle, v\rangle \models \varphi$ then we say that $\langle \mathcal{G}, \mu\rangle$ *satisfies* φ at v.

Of course, $\top$ is short-hand for $\neg\bot$ and the usual propositional connectives $\wedge$, $\vee$ and $\neg$ are definable in Hybrid Graph Logic, as are the dual modal operators $\Box$ and $\Box^+$, where $\Box\psi$ is short-hand for $\neg\Diamond\neg\psi$ and $\Box^+\psi$ is short-hand for $\neg\Diamond^+\neg\psi$; we shall use such abbreviations freely.

We now describe how we use Hybrid Graph Logic to define digraph problems; that is, classes of digraphs that are closed under isomorphisms.

Definition 4 Let φ be some formula of Hybrid Graph Logic that involves propositional symbols from the set P and nominals from the set N, and let $\mu : \mathsf{P} \cup \mathsf{N} \to \wp(V)$ be some valuation function, where $\mathcal{G} = \langle V, E\rangle$

is some frame. If $\langle\langle\mathcal{G}, \mu\rangle, v\rangle \models \varphi$, for every point v of $\mathcal{G}$, then we say that $\langle\mathcal{G}, \mu\rangle$ *globally satisfies* φ and write $\langle\mathcal{G}, \mu\rangle \models \varphi$. If every valuation function $\mu : \mathsf{P} \cup \mathsf{N} \to \wp(V)$ is such that $\langle\mathcal{G}, \mu\rangle \models \varphi$ then we say that φ is *valid* in $\mathcal{G}$ and write $\mathcal{G} \models \varphi$.

Definition 5 Let φ be some formula of Hybrid Graph Logic. The *problem* defined by φ consists of those frames $\mathcal{G}$ for which φ is valid in $\mathcal{G}$; that is, for which $\mathcal{G} \models \varphi$.

Care should be taken when working with formulae of Hybrid Graph Logic in relation to the problems they define. For example, consider some formula of HGL(P, N) of the form $\varphi \vee \psi$. We have that $\mathcal{G} = \langle V, E\rangle \models \varphi \vee \psi$ if, and only if, for all valuation functions $\mu : \mathsf{P} \cup \mathsf{N} \to \wp(V)$ and for all points u of $\mathcal{G}$, $\langle\langle\mathcal{G}, \mu\rangle, u\rangle \models \varphi \vee \psi$. However, we have that $\mathcal{G} \models \varphi$ or $\mathcal{G} \models \psi$ if, and only if, for all valuation functions $\mu : \mathsf{P} \cup \mathsf{N} \to \wp(V)$ and for all points u of $\mathcal{G}$, $\langle\langle\mathcal{G}, \mu\rangle, u\rangle \models \varphi$ or for all valuation functions $\mu : \mathsf{P} \cup \mathsf{N} \to \wp(V)$ and for all points u of $\mathcal{G}$, $\langle\langle\mathcal{G}, \mu\rangle, u\rangle \models \psi$. Thus, we might have that $\mathcal{G} \models \varphi \vee \psi$ but so that it is not necessarily the case that $\mathcal{G} \models \varphi$ or $\mathcal{G} \models \psi$.

Note that it might be argued that HGL is not very "well behaved" as a logic; for instance, it is not closed under negation (that is, the complementary problem of some problem defined by some formula φ of HGL need not be definable in HGL, where the *complementary problem* consists of all those frames $\mathcal{G}$ for which $\mathcal{G} \not\models \varphi$). However, there are many logics prevalent in descriptive complexity that are not closed under negation, existential second-order logic being perhaps the most prominent example!

Remark 6 Our notation differs from that used in [4]. Our frames are usually of the form $\mathcal{G} = \langle V, E\rangle$ (to reflect that we are dealing with *g*raphs, or more precisely di*g*raphs), whereas $\mathcal{F} = \langle W, R\rangle$ was the norm in [4]; our sets of *p*ropositional symbols are usually of the form P, whereas Φ was the norm in [4] (we reserve Greek letters for formulae, valuation functions and paths in digraphs); our sets of *n*ominals are usually of the form N, whereas Ψ was the norm in [4]; and our valuations are usually of the form μ, whereas $\mathbf{V}$ was the norm in [4].

2.2 Some problems definable in HGL

We now exhibit some problems definable in Hybrid Graph Logic. Two of these problems featured strongly in [4].

The problem STRONG-CONNECTIVITY consists of those digraphs for which there is a directed path from any vertex to any other vertex; that is, those digraphs that are strongly connected. This problem can be defined by the following formula φ of Hybrid Graph Logic:

$$\neg n \Rightarrow \Diamond^+ n$$

where n is a nominal.

To see this, suppose that the digraph $\mathcal{G} = \langle V, E\rangle$ is strongly connected but $\mathcal{G} \not\models \varphi$ (in particular, $\mathcal{G}$ has at least 2 vertices). There is a valuation function $\mu : \{n\} \rightarrow \wp(V)$, with $\mu(n) = \{v\}$, and a point u of $\mathcal{G}$ such that $\langle\langle \mathcal{G}, \mu\rangle, u\rangle \not\models \neg n \Rightarrow \Diamond^+ n$. So, $u \neq v$ and for every vertex v' of the digraph $\mathcal{G}$ reachable from u via a non-trivial path (that is, a path with at least one edge), we must have that $v' \neq v$. However, every vertex of $\mathcal{G}$ different from u (of which v is one) is reachable from u via a non-trivial path and so this yields a contradiction. Conversely, suppose that $\mathcal{G} \models \varphi$ but that the digraph $\mathcal{G}$ is not strongly connected. So, there exist distinct vertices u and v for which there is no path from u to v in the digraph $\mathcal{G}$. Define $\mu(n) = \{v\}$. In particular, $\langle\langle \mathcal{G}, \mu\rangle, u\rangle \models \neg n \Rightarrow \Diamond^+ n$. So, there is a path in the digraph $\mathcal{G}$ from u to v, which yields a contradiction.

The problem ACYCLIC consists of those digraphs that do not have a directed cycle containing at least one edge as a sub-digraph (note that we regard a self-loop as a cycle of length 1). This problem can be defined by the following formula φ of Hybrid Graph Logic:

$$@_n \neg \Diamond^+ n,$$

where n is a nominal.

To see this, suppose that the digraph $\mathcal{G} = \langle V, E\rangle$ is acyclic but $\mathcal{G} \not\models @_n \neg \Diamond^+ n$. So, there exists a valuation function $\mu : \{n\} \rightarrow \wp(V)$ and a point v of $\mathcal{G}$ such that $\langle\langle \mathcal{G}, \mu\rangle, v\rangle \not\models @_n \neg \Diamond^+ n$; that is, such that $\langle\langle \mathcal{G}, \mu\rangle, v'\rangle \not\models \neg \Diamond^+ n$, where $\mu(n) = v'$; that is, such that $\langle\langle \mathcal{G}, \mu\rangle, v'\rangle \models \Diamond^+ n$. Hence, there is a non-trivial path from v' to v' in the digraph $\mathcal{G}$ and we obtain a contradiction. Conversely, suppose that $\mathcal{G} \models @_n \neg \Diamond^+ n$ but the digraph $\mathcal{G}$ contains a cycle. Let v be a vertex on a cycle in $\mathcal{G}$ and define $\mu : \{n\} \rightarrow \wp(V)$ via $\mu(n) = \{v\}$. We have that $\langle\langle \mathcal{G}, \mu\rangle, v\rangle \models @_n \neg \Diamond^+ n$; that is, $\langle\langle \mathcal{G}, \mu\rangle, v\rangle \models \neg \Diamond^+ n$, which yields a contradiction.

The problem NON-4-COLSC consists of those digraphs that are strongly connected and can not be properly 4-coloured (we interpret the existence of a directed edge (u, v) in a digraph as meaning that u and v

must be coloured differently in any proper colouring). This problem can be defined by the following formula φ of Hybrid Graph Logic:

$$
\begin{aligned}
&(\neg n \Rightarrow \Diamond^+ n) \\
&\quad \wedge \Diamond^+((p_1 \wedge p_2 \wedge \Diamond(p_1 \wedge p_2)) \vee (p_1 \wedge \neg p_2 \wedge \Diamond(p_1 \wedge \neg p_2)) \\
&\qquad \vee(\neg p_1 \wedge p_2 \wedge \Diamond(\neg p_1 \wedge p_2)) \vee (\neg p_1 \wedge \neg p_2 \wedge \Diamond(\neg p_1 \wedge \neg p_2)))
\end{aligned}
$$

where n is a nominal and p_1 and p_2 are both propositional symbols.

To see this, we argue as follows. Let $\mathcal{G} = \langle V, E\rangle$ and let $\mu : \{p_1, p_2\} \cup \{n\} \to \wp(V)$ be any valuation function. We interpret μ as providing a 4-colouring of the vertices of V with the colours being: $v \in \mu(p_1)$ and $v \in \mu(p_2)$; $v \in \mu(p_1)$ and $v \notin \mu(p_2)$; $v \notin \mu(p_1)$ and $v \in \mu(p_2)$; and $v \notin \mu(p_1)$ and $v \notin \mu(p_2)$. Suppose that $\mathcal{G} \models \varphi$. So, $\mathcal{G}$ is strongly connected and for every valuation function $\mu : \{p_1, p_2\} \cup \{n\} \to \wp(V)$ (that is, every 4-colouring of the vertices of V) and for every $u \in V$, there exists a vertex v reachable via a non-trivial path from u so that there is an edge $(v, v') \in E$ where v and v' have the same colour. Hence, $\mathcal{G}$ is strongly connected and can not be properly 4-coloured. Conversely, if $\mathcal{G}$ is strongly connected and can not be properly 4-coloured then $\mathcal{G} \models \neg n \Rightarrow \Diamond^+ n$ and no matter which valuation function $\mu : \{p_1, p_2\} \cup \{n\} \to \wp(V)$ we choose and which point u of $\mathcal{G}$, there is a non-trivial path from u to a vertex v from which there is an edge (v, v') with v and v' identically coloured; that is, $\mathcal{G} \models \varphi$. This illustration shows that we can define problems in Hybrid Graph Logic that are probably not solvable in polynomial-time, as Non-4-ColSC is **co-NP**-complete, where **co-NP** is the class of problems whose complements are in **NP**. Note that it is immediate from the definition of what it means for a formula to be valid in some digraph that any problem defined be some formula of Hybrid Graph Logic is necessarily in **co-NP**.

2.3 Our research question

The logic HGL is basic enough that it forms a fragment of many other hybrid logics yet within it we can define computationally hard problems (assuming that $\mathbf{P} \neq \mathbf{NP}$); hence, it makes sense to undertake a fundamental study of this logic. As mentioned earlier, HGL is obtained from a well-studied fragment of PDL and CTL by allowing the use of nominals, and so it makes sense to better understand its expressibility, given that it will be a fragment of many well-known logics that are extended by the incorporation of nominals. Also, the study of the expressibility of 'low-level' logics in descriptive complexity is common-place; see, for example, the study of finite variable or quantifier-prefix fragments of first-order logic [8]. The logic HGL forms such a 'low-level' logic and is readily extendable to obtain new logics (as we shall mention in our conclusions), the expressibility of which will be built upon the expressibility of HGL.

The formulation of formulae defining problems such as STRONG-CONNECTIVITY, ACYCLICITY and NON-4-COLSC leads us to pose the main question studied in this paper:

'*Can we determine the relative definability of fragments of Hybrid Graph Logic obtained by restricting*: *the application of the operators* $\Diamond$, $\Diamond^+$, $\Box$, $\Box^+$ *and* @; *the number of propositional symbols used*; *and the number of nominals used*?'

In order to make this question more precise, we need to define what we mean by the quantifier-rank of a formula of Hybrid Graph Logic.

Definition 7 Every atomic formula φ has quantifier-rank 0 and we write $qr(\varphi) = 0$ for such formulae.

- If φ is of the form $\neg\psi$ then φ has quantifier-rank defined as $qr(\varphi) = qr(\psi)$.
- If φ is of the form $\psi \Rightarrow \psi'$, $\psi \wedge \psi'$ or $\psi \vee \psi'$ then φ has quantifier-rank defined as $qr(\varphi) = \max\{qr(\psi), qr(\psi')\}$.
- If φ is of the form $\Diamond\psi$, $\Diamond^+\psi$, $\Box\psi$, $\Box^+\psi$ or $@_n\psi$ then φ has quantifier-rank defined as $qr(\psi) + 1$.

Definition 8 Let c, d and r be non-negative integers, let P be a set of propositional symbols and let N be a set of nominals. The logic $\mathrm{HGL}_r(\mathsf{P}, \mathsf{N})$ consists of those formulae of $\mathrm{HGL}(\mathsf{P}, \mathsf{N})$ of quantifier-rank at most r. The logic $\mathrm{HGL}_r(c, d)$ is the fragment of Hybrid Graph Logic consisting of all those formulae of quantifier-rank at most r in which there are at most c propositional symbols and d nominals.

(Note that just as for HGL, we blur the distinction between language and logic for $\mathrm{HGL}_r(c, d)$.) We equate the logic $\mathrm{HGL}_r(c, d)$ with the class of problems definable by the formulae of that logic; consequently, we write, for example, $\mathrm{HGL}_r(c, d) \subseteq \mathrm{HGL}_{r'}(c', d')$ to denote that any problem definable by a formula of the logic $\mathrm{HGL}_r(c, d)$ can also be defined by a formula of the logic $\mathrm{HGL}_{r'}(c', d')$, and we write $\mathrm{HGL}_r(c, d) \subset \mathrm{HGL}_{r'}(c', d')$

to denote that $\mathrm{HGL}_r(c,d) \subseteq \mathrm{HGL}_{r'}(c',d')$ but there are problems definable in $\mathrm{HGL}_{r'}(c',d')$ that are not definable in $\mathrm{HGL}_r(c,d)$. Thus, we have that STRONG-CONNECTIVITY $\in \mathrm{HGL}_1(0,1)$, ACYCLIC $\in \mathrm{HGL}_2(0,1)$ and NON-4-COLSC $\in \mathrm{HGL}_2(2,1)$, with $\mathrm{HGL}_1(0,1) \subseteq \mathrm{HGL}_2(0,2) \subseteq \mathrm{HGL}_2(2,2)$.

Related to the logic $\mathrm{HGL}_r(c,d)$ are various notions of equivalence.

Definition 9 Let P be a set of propositional symbols and let N be a set of nominals.

- Let $\langle\langle \mathcal{G}, \mu\rangle, u\rangle$ and $\langle\langle \mathcal{H}, \lambda\rangle, v\rangle$ be pointed Kripke $\mathsf{P} \cup \mathsf{N}$-structures. We say that $\langle\langle \mathcal{G}, \mu\rangle, u\rangle$ and $\langle\langle \mathcal{H}, \lambda\rangle, v\rangle$ are $(\mathsf{P}, \mathsf{N}, r)$-*equivalent*, and write $\langle\langle \mathcal{G}, \mu\rangle, u\rangle \equiv_{\mathrm{HGL}_r(\mathsf{P},\mathsf{N})} \langle\langle \mathcal{H}, \lambda\rangle, v\rangle$, if for all formulae φ of $\mathrm{HGL}_r(\mathsf{P}, \mathsf{N})$, $\langle\langle \mathcal{G}, \mu\rangle, u\rangle \models \varphi$ if, and only if, $\langle\langle \mathcal{H}, \lambda\rangle, v\rangle \models \varphi$.

- The Kripke $\mathsf{P} \cup \mathsf{N}$-structures $\langle \mathcal{G}, \mu\rangle$ and $\langle \mathcal{H}, \lambda\rangle$ are $(\mathsf{P}, \mathsf{N}, r)$-*equivalent*, and we write $\langle \mathcal{G}, \mu\rangle \equiv_{\mathrm{HGL}_r(\mathsf{P},\mathsf{N})} \langle \mathcal{H}, \lambda\rangle$, if $\langle\langle \mathcal{G}, \mu\rangle, u\rangle \equiv_{\mathrm{HGL}_r(\mathsf{P},\mathsf{N})} \langle\langle \mathcal{H}, \lambda\rangle, v\rangle$, for all points u and v of $\mathcal{G}$ and $\mathcal{H}$, respectively.

- The frames $\mathcal{G}$ and $\mathcal{H}$ are (c, d, r)-*equivalent*, and we write $\mathcal{G} \equiv_{\mathrm{HGL}_r(c,d)} \mathcal{H}$, if $\mathcal{G} \models \varphi$ if, and only if, $\mathcal{H} \models \varphi$, for all formulae φ of $\mathrm{HGL}_r(c,d)$. There is an analogous definition of two frames being $(\mathsf{P}, \mathsf{N}, r)$-*equivalent*.

Note that the restriction of formulae of modal logics by restricting the quantifier depth or the number of propositional symbols employed is not new; certainly, such restrictions were made in [12] to various modal logics in the context of the complexity of the satisfiability problem in these logics.

2.4 Some loosely related results

Whilst HGL has undergone limited investigation in the context of defining isomorphism-closed classes of finite frames, there are some existing results that are loosely related to our propose research and provide a bit more motivation for our studies.

As was noted in [4], results from [2] and [10], respectively, show that the satisfiability and validity problems for HGL are **EXPTIME**-complete and the model-checking problem for HGL is solvable in time that is linear in the size of the model and the length of the formula.

As an example of the study of general hybrid logic expressibility, Areces, Blackburn and Marx [3] studied the hybrid logic $\mathcal{H}(\downarrow, @)$, which is the extension of propositional modal logic with the binder $\downarrow$ and nominal operators of the form $@_n$, and characterized its expressibility (on Kripke structures) in terms of a fragment of first-order logic (namely the fragment that is invariant under generated sub-models). They developed both Ehrenfeucht-Fraïssé-style games and notions of bisimulation in doing so. Their investigation is primarily as regards Kripke structures although they briefly mention the relevance of their work to frames. They mention the study of the hybrid logic $\mathcal{H}(@)$, in relation to its expressibility, as being an interesting direction for further research. Note that HGL is more expressive than $\mathcal{H}(@)$.

3 Games and Hybrid Graph Logic

In this section, we develop Ehrenfeucht-Fraïssé-style pebble games to capture definability in fragments of Hybrid Graph Logic. We go on to use these games in the next section to establish strict hierarchy results within Hybrid Graph Logic.

3.1 Games on pointed Kripke structures

We begin by playing games on pointed Kripke structures.

Definition 10 Let $r \geq 0$, let P be a set of propositional symbols and let N be a set of nominals. The *r-round HGL(P, N)-game* is a two-player pebble game played by Spoiler and Duplicator. It is played on two pointed Kripke $\mathsf{P} \cup \mathsf{N}$-structures $\langle\langle\mathcal{G}, \mu\rangle, u\rangle$ and $\langle\langle\mathcal{H}, \lambda\rangle, v\rangle$. Initially, prior to any play of the game, pebble a sits on point u of $\mathcal{G}$ and pebble b sits on point v of $\mathcal{H}$. During a play of the game, pebble a is moved from point to point in $\mathcal{G}$, with pebble b moved from point to point in $\mathcal{H}$. There are r rounds in any play of the game and we denote the point of $\mathcal{G}$ (resp. $\mathcal{H}$) on which pebble a (resp. b) sits after round i of some play by a_i (resp. b_i); so, $a_0 = u$ and $b_0 = v$. Each round i of a play is as follows.

1. Spoiler makes one of the following moves.

   (a) $\Diamond$-move: Spoiler moves pebble a from a_{i-1} to a_i. It must be the case that there is an edge (a_{i-1}, a_i) in the digraph $\mathcal{G}$.

(b) $\Box$-move: Spoiler moves pebble b from b_{i-1} to b_i. It must be the case that there is an edge (b_{i-1}, b_i) in the digraph $\mathcal{H}$.

(c) $\Diamond^+$-move: Spoiler moves pebble a from a_{i-1} to a_i. It must be the case that there is a non-trivial path (that is, consisting of at least one edge, though possibly a self-loop) from a_{i-1} to a_i in the digraph $\mathcal{G}$.

(d) $\Box^+$-move: Spoiler moves pebble b from b_{i-1} to b_i. It must be the case that there is a non-trivial path from b_{i-1} to b_i in the digraph $\mathcal{H}$.

(e) $@_n$-move: Spoiler moves the pebble a to the point $\mu(n)$ (note that this move has no dual).

2. Duplicator replies with the corresponding move.

   (a) $\Diamond$-move: Duplicator moves pebble b from b_{i-1} to b_i. It must be the case that there is an edge (b_{i-1}, b_i) in the digraph $\mathcal{H}$.

   (b) $\Box$-move: Duplicator moves pebble a from a_{i-1} to a_i. It must be the case that there is an edge (a_{i-1}, a_i) in the digraph $\mathcal{G}$.

   (c) $\Diamond^+$-move: Duplicator moves pebble b from b_{i-1} to b_i. It must be the case that there is a non-trivial path from b_{i-1} to b_i in the digraph $\mathcal{H}$.

   (d) $\Box^+$-move: Duplicator moves pebble a from a_{i-1} to a_i. It must be the case that there is a non-trivial path from a_{i-1} to a_i in the digraph $\mathcal{G}$.

   (e) $@_n$-move: Duplicator moves the pebble b to the point $\lambda(n)$.

In order to determine the winner of a play of a game, we need to define a notion of equivalence on points.

Definition 11 Let $\langle \mathcal{G}, \mu \rangle$ and $\langle \mathcal{H}, \lambda \rangle$ be two Kripke $\mathsf{P} \cup \mathsf{N}$-structures, and let u and v be points of $\mathcal{G}$ and $\mathcal{H}$, respectively. We say that u and v are *equivalent*, and write $u \simeq v$, if for every $p \in \mathsf{P}$ and $n \in \mathsf{N}$: $u \in \mu(p)$ if, and only if, $v \in \lambda(p)$; and $u = \mu(n)$ if, and only if, $v = \lambda(n)$.

Note that we might have that $\mathcal{G} = \mathcal{H}$.

Definition 12 Duplicator wins a play of the r-round HGL(P, N)-game on two pointed Kripke $\mathsf{P} \cup \mathsf{N}$-structures $\langle \langle \mathcal{G}, \mu \rangle, u \rangle$ and $\langle \langle \mathcal{H}, \lambda \rangle, v \rangle$ if either:

at some point before the end of the play Spoiler cannot make a move (which is only the case when both pebbles are on vertices of out-degree 0 and there are no nominals); or for all $i \in \{0, 1, \ldots, r\}$, $a_i \simeq b_i$. In all other cases, Spoiler wins; in particular, Spoiler wins a play if Duplicator is unable to reply to a move of Spoiler. Duplicator has a winning strategy in the r-round HGL(P, N)-game on two pointed Kripke $\mathsf{P} \cup \mathsf{N}$-structures if she can always reply to Spoiler's moves in any play so as to force a win for Duplicator, with Spoiler having a winning strategy if he can move so as to force at least one play of the game for which Duplicator does not win.

There is a relationship between r-round HGL(P, N)-games and pointed Kripke structure definability in the logic $\mathrm{HGL}_r(\mathsf{P}, \mathsf{N})$.

Theorem 13 Let $r \geq 0$, let P be a set of propositional symbols and let N be a set of nominals. Let $\langle \mathcal{G}, \mu \rangle$ and $\langle \mathcal{H}, \lambda \rangle$ be Kripke $\mathsf{P} \cup \mathsf{N}$-structures and let u and v be points of $\mathcal{G}$ and $\mathcal{H}$, respectively. The following are equivalent.

1. $\langle \langle \mathcal{G}, \mu \rangle, u \rangle \equiv_{\mathrm{HGL}_r(\mathsf{P},\mathsf{N})} \langle \langle \mathcal{H}, \lambda \rangle, v \rangle$.
2. Duplicator has a winning strategy in the r-round HGL(P, N)-game on $\langle \langle \mathcal{G}, \mu \rangle, u \rangle$ and $\langle \langle \mathcal{H}, \lambda \rangle, v \rangle$.

Proof For brevity, throughout this proof: on occasion we write $\mathcal{C} = \langle \mathcal{G}, \mu \rangle$ and $\mathcal{D} = \langle \mathcal{H}, \lambda \rangle$; we denote the edge set of the digraph $\mathcal{G}$ (resp. $\mathcal{H}$) by $E^{\mathcal{G}}$ (resp. $E^{\mathcal{H}}$); and we denote the transitive closure of $E^{\mathcal{G}}$ (resp. $E^{\mathcal{H}}$) by $E^{\mathcal{G}}_{+}$ (resp. $E^{\mathcal{H}}_{+}$).

We begin by noting that our result trivially holds for $r = 0$. In particular, for any point u of some Kripke $\mathsf{P} \cup \mathsf{N}$-structure $\langle \mathcal{G}, \mu \rangle$, there exists a formula $\varphi^0_{\mathcal{C},u}$ of $\mathrm{HGL}_0(\mathsf{P}, \mathsf{N})$ such that for any point v of any Kripke $\mathsf{P} \cup \mathsf{N}$-structure $\langle \mathcal{H}, \lambda \rangle$, $\langle \langle \mathcal{G}, \mu \rangle, u \rangle \equiv_{\mathrm{HGL}_0(\mathsf{P},\mathsf{N})} \langle \langle \mathcal{H}, \lambda \rangle, v \rangle$ if, and only if, $u \simeq v$ if, and only if, Duplicator has a winning strategy in the 0-round HGL(P, N)-game on $\langle \langle \mathcal{G}, \mu \rangle, u \rangle$ and $\langle \langle \mathcal{H}, \lambda \rangle, v \rangle$ if, and only if, $\langle \langle \mathcal{H}, \lambda \rangle, v \rangle \models \varphi^0_{\mathcal{C},u}$. The formula $\varphi^0_{\mathcal{C},u}$ is actually the formula

$$\bigwedge\{p : p \in \mathsf{P}, u \in \mu(p)\} \wedge \bigwedge\{\neg p : p \in \mathsf{P}, u \notin \mu(p)\} \\ \wedge \bigwedge\{n : n \in \mathsf{N}, u = \mu(n)\} \wedge \bigwedge\{\neg n : n \in \mathsf{N}, u \neq \mu(n)\}.$$

Lemma 14 Let $r \geq 0$ and let $\langle \mathcal{G}, \mu \rangle$ be a Kripke $\mathsf{P} \cup \mathsf{N}$-structure. For any point u of $\mathcal{G}$, there exists a formula $\varphi^r_{\mathcal{C},u}$ of $\mathrm{HGL}_r(\mathsf{P}, \mathsf{N})$ such that for any point v of any Kripke $\mathsf{P} \cup \mathsf{N}$-structure $\langle \mathcal{H}, \lambda \rangle$, Duplicator has a winning strategy in the r-round $\mathrm{HGL}(\mathsf{P}, \mathsf{N})$-game on $\langle \langle \mathcal{G}, \mu \rangle, u \rangle$ and $\langle \langle \mathcal{H}, \lambda \rangle, v \rangle$ if, and only if, $\langle \langle \mathcal{H}, \lambda \rangle, v \rangle \models \varphi^r_{\mathcal{C},u}$.

Proof Let the following be our induction hypothesis $\mathrm{Ind}(r)$, for some $r \geq 0$: for any point u of $\mathcal{G}$, there exists a formula $\varphi^r_{\mathcal{C},u}$ of $\mathrm{HGL}_r(\mathsf{P}, \mathsf{N})$ such that for any point v of any Kripke $\mathsf{P} \cup \mathsf{N}$-structure $\langle \mathcal{H}, \lambda \rangle$, Duplicator has a winning strategy in the r-round $\mathrm{HGL}(\mathsf{P}, \mathsf{N})$-game on $\langle \mathcal{C}, u \rangle$ and $\langle \mathcal{D}, v \rangle$ if, and only if, $\langle \mathcal{D}, v \rangle \models \varphi^r_{\mathcal{C},u}$. $\mathrm{Ind}(0)$ holds by the remark preceding the statement of the lemma.

For any point u of $\mathcal{G}$, define the formula $\varphi^{r+1}_{\mathcal{C},u}$ of $\mathrm{HGL}_{r+1}(\mathsf{P}, \mathsf{N})$ as follows:

$$\bigwedge_{(u,u') \in E^{\mathcal{G}}} \Diamond \varphi^r_{\mathcal{C},u'} \wedge \bigwedge_{(u,u') \in E^{\mathcal{G}}_+} \Diamond^+ \varphi^r_{\mathcal{C},u'} \wedge \Box \bigvee_{(u,u') \in E^{\mathcal{G}}} \varphi^r_{\mathcal{C},u'} \wedge \Box^+ \bigvee_{(u,u') \in E^{\mathcal{G}}_+} \varphi^r_{\mathcal{C},u'}$$
$$\wedge \bigwedge_{n \in \mathsf{N}} @_n \varphi^r_{\mathcal{C},\mu(n)}.$$

Suppose that: v is some point of some Kripke $\mathsf{P} \cup \mathsf{N}$-structure $\langle \mathcal{H}, \lambda \rangle$; $\langle \mathcal{D}, v \rangle \models \varphi^{r+1}_{\mathcal{C},u}$; and we play an $(r+1)$-round $\mathrm{HGL}(\mathsf{P}, \mathsf{N})$-game on $\langle \mathcal{C}, u \rangle$ and $\langle \mathcal{D}, v \rangle$.

1. Suppose that Spoiler's first move in some play is a $\Diamond$-move so that $a_1 = u'$. As $\langle \mathcal{D}, v \rangle \models \varphi^{r+1}_{\mathcal{C},u}$, we have that $\langle \mathcal{D}, v \rangle \models \Diamond \varphi^r_{\mathcal{C},u'}$. So, there exists some point $v' \in \{v' \text{ is a point of } \mathcal{H} : (v, v') \in E^{\mathcal{H}}\}$ such that $\langle \mathcal{D}, v' \rangle \models \varphi^r_{\mathcal{C},u'}$. Duplicator replies with $b_1 = v'$. By $\mathrm{Ind}(r)$, Duplicator has a winning strategy in the r-round $\mathrm{HGL}(\mathsf{P}, \mathsf{N})$-game on $\langle \mathcal{C}, u' \rangle$ and $\langle \mathcal{D}, v' \rangle$. If Spoiler's move is a $\Diamond^+$-move then we argue similarly.

2. Suppose that Spoiler's first move in some play is a $\Box$-move so that $b_1 = v'$. As $\langle \mathcal{D}, v \rangle \models \varphi^{r+1}_{\mathcal{C},u}$, we have that $\langle \mathcal{D}, v \rangle \models \Box \bigvee_{(u,u') \in E^{\mathcal{G}}} \varphi^r_{\mathcal{C},u'}$; that is, $\langle \mathcal{D}, v' \rangle \models \bigvee_{(u,u') \in E^{\mathcal{G}}} \varphi^r_{\mathcal{C},u'}$. Hence, $\langle \mathcal{D}, v' \rangle \models \varphi^r_{\mathcal{C},u'}$, for some point $u' \in \{u' \text{ is a point of } \mathcal{G} : (u, u') \in E^{\mathcal{G}}\}$. Duplicator replies with $a_1 = u'$. By $\mathrm{Ind}(r)$, Duplicator has a winning strategy in the r-round $\mathrm{HGL}(\mathsf{P}, \mathsf{N})$-game on $\langle \mathcal{C}, u' \rangle$ and $\langle \mathcal{D}, v' \rangle$. If Spoiler's move is a $\Box^+$-move then we argue similarly.

3. Suppose that Spoiler's first move in some play is a $@_n$-move so that $a_1 = \mu(n)$. As $\langle \mathcal{D}, v\rangle \models \varphi^{r+1}_{\mathcal{C},u}$, we have that $\langle \mathcal{D}, v\rangle \models @_n\varphi^r_{\mathcal{C},\mu(n)}$; that is, $\langle \mathcal{D}, \lambda(n)\rangle \models \varphi^r_{\mathcal{C},\mu(n)}$. Duplicator replies with $b_1 = \lambda(n)$. By Ind(r), Duplicator has a winning strategy in the r-round HGL(P, N)-game on $\langle \mathcal{C}, \mu(n)\rangle$ and $\langle \mathcal{D}, \lambda(n)\rangle$.

Consequently, by replying with a_1 or b_1, appropriately, as in each of the cases above, Duplicator has a winning strategy in the $(r + 1)$-round HGL(P, N)-game on $\langle \mathcal{C}, u\rangle$ and $\langle \mathcal{D}, v\rangle$.

Conversely, suppose that v is a point of some Kripke $\mathsf{P} \cup \mathsf{N}$-structure $\langle \mathcal{H}, \lambda\rangle$ and that Duplicator has a winning strategy in the $(r + 1)$-round HGL(P, N)-game on $\langle \mathcal{C}, u\rangle$ and $\langle \mathcal{D}, v\rangle$. Consider a play of the game.

1. Suppose that Spoiler's first move in some play is a $\Diamond$-move so that $a_1 = u'$. Let Duplicator's response (according to the winning strategy) be $b_1 = v'$. In particular, Duplicator has a winning strategy in the r-round HGL(P, N)-game on $\langle \mathcal{C}, u'\rangle$ and $\langle \mathcal{D}, v'\rangle$. By Ind($r$), $\langle \mathcal{D}, v'\rangle \models \varphi^r_{\mathcal{C},u'}$ and so $\langle \mathcal{D}, v\rangle \models \Diamond\varphi^r_{\mathcal{C},u'}$. Hence, as Spoiler's first $\Diamond$-move can be arbitrary amongst all points of $\{u'$ is a point of $\mathcal{G}$: $(u, u') \in E^{\mathcal{G}}\}$, we have that $\langle \mathcal{D}, v\rangle \models \bigwedge_{(u,u')\in E^{\mathcal{G}}} \Diamond\varphi^r_{\mathcal{C},u'}$. Similar reasoning yields that $\langle \mathcal{D}, v\rangle \models \bigwedge_{(u,u')\in E^{\mathcal{G}}_+} \Diamond^+\varphi^r_{\mathcal{C},u'}$.

2. Suppose that Spoiler's first move in some play is a $\Box$-move. No matter which point of $\{v'$ is a point of $\mathcal{H} : (v, v') \in E^{\mathcal{H}}\}$ Spoiler moves to, so that $b_1 = v'$, there exists some point $g(v')$ of $\{u'$ is a point of $\mathcal{G} : (u, u') \in E^{\mathcal{G}}\}$ that Duplicator can respond with, by setting $a_1 = g(v')$, so as to ensure that Duplicator wins the subsequent r-round HGL(P, N)-game on $\langle \mathcal{C}, g(v')\rangle$ and $\langle \mathcal{D}, v'\rangle$. By Ind($r$), $\langle \mathcal{D}, v'\rangle \models \varphi^r_{\mathcal{C},g(v')}$, for every point v' in $\{v'$ is a point of $\mathcal{H} : (v, v') \in E^{\mathcal{H}}\}$. Consequently, we have that $\langle \mathcal{D}, v\rangle \models \Box \bigvee_{(u,u')\in E^{\mathcal{G}}} \varphi^r_{\mathcal{C},u'}$. Similar reasoning yields that $\langle \mathcal{D}, v\rangle \models \Box^+ \bigvee_{(u,u')\in E^{\mathcal{G}}_+} \varphi^r_{\mathcal{C},u'}$.

3. Suppose that Spoiler's first move in some play is a $@_n$-move, so that $a_1 = \mu(n)$. Duplicator's reply must be so that $b_1 = \lambda(n)$, and Duplicator wins the subsequent r-round HGL(P, N)-game on $\langle \mathcal{C}, \mu(n)\rangle$ and $\langle \mathcal{D}, \lambda(n)\rangle$. By Ind($r$), $\langle \mathcal{D}, \lambda(n)\rangle \models \varphi^r_{\mathcal{C},\mu(n)}$ and so $\langle \mathcal{D}, v\rangle \models @_n\varphi^r_{\mathcal{C},\mu(n)}$; consequently, as n is an arbitrary nominal, we have that $\langle \mathcal{D}, v\rangle \models \bigwedge_{n\in\mathsf{N}} @_n\varphi^r_{\mathcal{C},\mu(n)}$.

Hence, $\langle \mathcal{D}, v\rangle \models \varphi^{r+1}_{\mathcal{C},u}$, and the lemma follows by induction. $\square$

We can now return to our proof of the main theorem. As our induction hypothesis Ind(r), where $r \geq 0$, we assume that for any Kripke $\mathsf{P} \cup \mathsf{N}$-structures $\langle \mathcal{G}, \mu \rangle$ and $\langle \mathcal{H}, \lambda \rangle$ and for any points u and v of $\mathcal{G}$ and $\mathcal{H}$, respectively, the following are equivalent:

1. $\langle \langle \mathcal{G}, \mu \rangle, u \rangle \equiv_{\mathrm{HGL}_r(\mathsf{P},\mathsf{N})} \langle \langle \mathcal{H}, \lambda \rangle, v \rangle$.

2. Duplicator has a winning strategy in the r-round HGL(P, N)-game on $\langle \langle \mathcal{G}, \mu \rangle, u \rangle$ and $\langle \langle \mathcal{H}, \lambda \rangle, v \rangle$.

We have already remarked that Ind(0) holds.

Let $\langle \mathcal{G}, \mu \rangle$ and $\langle \mathcal{H}, \lambda \rangle$ be Kripke $\mathsf{P} \cup \mathsf{N}$-structures and u and v points of $\mathcal{G}$ and $\mathcal{H}$, respectively. Suppose that $\langle \mathcal{C}, u \rangle \equiv_{\mathrm{HGL}_{r+1}(\mathsf{P},\mathsf{N})} \langle \mathcal{D}, v \rangle$; that is, for every formula $\varphi \in \mathrm{HGL}_{r+1}(\mathsf{P}, \mathsf{N})$, $\langle \mathcal{C}, u \rangle \models \varphi$ if, and only if, $\langle \mathcal{D}, v \rangle \models \varphi$. Let us assume that Spoiler has a winning strategy for the $(r + 1)$-round HGL(P, N)-game played on the Kripke $\mathsf{P} \cup \mathsf{N}$-structures $\langle \mathcal{C}, u \rangle$ and $\langle \mathcal{D}, v \rangle$. By (two applications of) Lemma 14, $\langle \mathcal{C}, u \rangle \models \varphi^{r+1}_{\mathcal{C},u}$ and $\langle \mathcal{D}, v \rangle \not\models \varphi^{r+1}_{\mathcal{C},u}$, with $\varphi^{r+1}_{\mathcal{C},u} \in \mathrm{HGL}_{r+1}(\mathsf{P}, \mathsf{N})$. This yields a contradiction.

Conversely, suppose that $\langle \mathcal{C}, u \rangle \not\equiv_{\mathrm{HGL}_{r+1}(\mathsf{P},\mathsf{N})} \langle \mathcal{D}, v \rangle$. So, there exists a formula $\varphi \in \mathrm{HGL}_{r+1}(\mathsf{P}, \mathsf{N})$ such that $\langle \mathcal{C}, u \rangle \models \varphi$ and $\langle \mathcal{D}, v \rangle \not\models \varphi$. W.l.o.g. we may assume that φ is of the form

$$\Diamond\psi \ ; \ \Diamond^+\psi \ ; \ \Box\psi \ ; \ \Box^+\psi \ ; \text{ or } @_n\psi,$$

where $\psi \in \mathrm{HGL}_r(\mathsf{P}, \mathsf{N})$ and n is a nominal of N. To see this, note that if φ is of the form $\varphi_1 \Rightarrow \varphi_2$, for example, then $\langle \mathcal{D}, u \rangle \models \varphi_1$ but $\langle \mathcal{D}, u \rangle \not\models \varphi_2$. If $\langle \mathcal{C}, u \rangle \models \varphi_1$ then take φ as φ_2, otherwise take φ as $\neg\varphi_1$.

1. Suppose that φ is of the form $\Diamond\psi$. There exists a point u' of $\mathcal{G}$ such that $(u, u') \in E^{\mathcal{G}}$ and $\langle \mathcal{C}, u' \rangle \models \psi$. Let Spoiler's first move of an $(r+1)$-round HGL(P, N)-game on $\langle \mathcal{C}, u \rangle$ and $\langle \mathcal{D}, v \rangle$ be a $\Diamond$-move so that $a_1 = u'$. No matter which point v' of $\{v'$ is a point of $\mathcal{H} : (v, v') \in E^{\mathcal{H}}\}$ Duplicator replies with, $\langle \mathcal{D}, v' \rangle \not\models \psi$. By Ind($r$), Spoiler has a winning strategy in the subsequent r-round HGL(P, N)-game on $\langle \mathcal{C}, u' \rangle$ and $\langle \mathcal{D}, v' \rangle$. We argue similarly when φ is of the form $\Diamond^+\psi$.

2. Suppose that φ is of the form $\Box\psi$. For every point u' of $\mathcal{G}$ such that $(u, u') \in E^{\mathcal{G}}$, we have that $\langle \mathcal{C}, u' \rangle \models \psi$. However, there is a point v' of $\{v'$ is a point of $\mathcal{H} : (v, v') \in E^{\mathcal{H}}\}$ such that $\langle \mathcal{D}, v' \rangle \not\models \psi$. Let Spoiler's first move of an $(r + 1)$-round HGL(P, N)-game be a $\Box$-move so that $b_1 = v'$. No matter which point u' of $\{u'$ is a point of

$\mathcal{G} : (u, u') \in E^{\mathcal{G}}\}$ Duplicator replies with, $\langle \mathcal{C}, u' \rangle \models \psi$. By Ind($r$), Spoiler has a winning strategy in the subsequent r-round HGL(P, N)-game on $\langle \mathcal{C}, u' \rangle$ and $\langle \mathcal{D}, v' \rangle$. We argue similarly when φ is of the form $\Box^+\psi$.

3. Suppose that φ is of the form $@_n\psi$, where n is a nominal of N. Thus, $\langle \mathcal{C}, \mu(n) \rangle \models \psi$ and $\langle \mathcal{D}, \lambda(n) \rangle \not\models \psi$. Let Spoiler's first move of an $(r+1)$-round HGL(P, N)-game be an $@_n$-move. By Ind(r), Spoiler has a winning strategy in the subsequent r-round HGL(P, N)-game on $\langle \mathcal{C}, \mu(n) \rangle$ and $\langle \mathcal{D}, \lambda(n) \rangle$.

Thus, Spoiler has a winning strategy in the $(r + 1)$-round HGL(P, N)-game played on the Kripke $\mathsf{P} \cup \mathsf{N}$-structures $\langle \langle \mathcal{G}, \mu \rangle, u \rangle$ and $\langle \langle \mathcal{H}, \lambda \rangle, v \rangle$. The result follows by induction. □

3.2 Games on frames

We now extend the games of the previous sub-section to games on frames as opposed to pointed Kripke structures.

Definition 15 Let P be a set of propositional symbols and let N be a set of nominals. The *r-round colouring HGL*(P, N)*-game* is played on two frames $\mathcal{G}$ and $\mathcal{H}$ and proceeds as follows.

- Either:

  (a) Spoiler chooses a valuation function $\lambda : \mathsf{P} \cup \mathsf{N} \to \wp(V^{\mathcal{H}})$ and a point v of $\mathcal{H}$

  or:

  (b) Spoiler chooses a valuation function $\mu : \mathsf{P} \cup \mathsf{N} \to \wp(V^{\mathcal{G}})$ and a point u of $\mathcal{G}$.

- In case (a):

  – Duplicator replies with a valuation function $\mu : \mathsf{P} \cup \mathsf{N} \to \wp(V^{\mathcal{G}})$ and a point u of $\mathcal{G}$

  and in case (b):

  – Duplicator replies with a valuation function $\lambda : \mathsf{P} \cup \mathsf{N} \to \wp(V^{\mathcal{H}})$ and a point v of $\mathcal{H}$.

- There then follows an r-round HGL(P, N)-game on the pointed $\mathsf{P} \cup \mathsf{N}$-structures $\langle\langle \mathcal{G}, \mu\rangle, u\rangle$ and $\langle\langle \mathcal{H}, \lambda\rangle, v\rangle$.

The winning conditions for a play of the r-round colouring HGL(P, N)-game are analogous to those of the r-round HGL(P, N)-game.

We refer to the choice of pointed Kripke structures (on which to play the subsequent r-round HGL(P, N)-game) by Spoiler and Duplicator as the *colouring-start phase* of a play of the game, with the choice made by either Spoiler or Duplicator as their *colouring-start move*.

There is a relationship between r-round colouring HGL(P, N)-games and frame definability in the logic $\mathrm{HGL}_r(\mathsf{P}, \mathsf{N})$.

Theorem 16 Let $r \geq 0$, let P be a set of propositional symbols and let N be a set of nominals. Let $\mathcal{G}$ and $\mathcal{H}$ be two frames. The following are equivalent.

1. $\mathcal{G} \equiv_{\mathrm{HGL}_r(\mathsf{P},\mathsf{N})} \mathcal{H}$.
2. Duplicator has a winning strategy in the r-round colouring HGL(P, N)-game on $\mathcal{G}$ and $\mathcal{H}$.

Proof Assume that $\mathcal{G} \equiv_{\mathrm{HGL}_r(\mathsf{P},\mathsf{N})} \mathcal{H}$. Suppose that Spoiler has a winning strategy in the r-round colouring HGL(P, N)-game on $\mathcal{G}$ and $\mathcal{H}$, and consider the following play. According to his winning strategy, Spoiler builds the Kripke $\mathsf{P} \cup \mathsf{N}$-structure $\langle \mathcal{H}, \lambda\rangle$ and chooses a point v of $\mathcal{H}$. Duplicator responds by building some Kripke $\mathsf{P} \cup \mathsf{N}$-structure $\langle \mathcal{G}, \mu\rangle$ and choosing the point u of $\mathcal{G}$. Irrespective of Duplicator's choice, Spoiler can now win the r-round HGL(P, N)-game on $\langle\langle \mathcal{G}, \mu\rangle, u\rangle$ and $\langle\langle \mathcal{H}, \lambda\rangle, v\rangle$. Thus, by Theorem 13, there is a formula $\varphi^r_{\mu,u}$ of $\mathrm{HGL}_r(\mathsf{P}, \mathsf{N})$ for which $\langle\langle \mathcal{G}, \mu\rangle, u\rangle \models \varphi^r_{\mu,u}$ and $\langle\langle \mathcal{H}, \lambda\rangle, v\rangle \not\models \varphi^r_{\mu,u}$. Define Φ as $\bigvee\{\varphi^r_{\mu,u} : \mu : \mathsf{P} \cup \mathsf{N} \to \wp(V^{\mathcal{G}})$ is a valuation function, u is a point of $\mathcal{G}\}$. In particular, $\mathcal{G} \models \Phi$, and so, by hypothesis, we must have that $\mathcal{H} \models \Phi$. However, $\langle\langle \mathcal{H}, \lambda\rangle, v\rangle \not\models \Phi$, which yields a contradiction. There is an analogous argument should Spoiler first build the Kripke $\mathsf{P} \cup \mathsf{N}$-structure $\langle \mathcal{H}, \lambda\rangle$ and choose the point v of $\mathcal{H}$.

Conversely, suppose that there exists a formula φ of $\mathrm{HGL}_r(\mathsf{P}, \mathsf{N})$ such that either $\mathcal{G} \models \varphi$ and $\mathcal{H} \not\models \varphi$ or $\mathcal{G} \not\models \varphi$ and $\mathcal{H} \models \varphi$. Suppose it is the former case. In particular, there exist some valuation function $\lambda : \mathsf{P} \cup \mathsf{N} \to \wp(V^{\mathcal{H}})$ and some point v of $\mathcal{H}$ such that $\langle\langle \mathcal{H}, \lambda\rangle, v\rangle \not\models \varphi$. Spoiler begins by building the Kripke $\mathsf{P} \cup \mathsf{N}$-structure $\langle \mathcal{H}, \lambda\rangle$ and chooses the point v of $\mathcal{H}$. No matter how Duplicator replies, by building some Kripke $\mathsf{P}\cup\mathsf{N}$-structure $\langle\mathcal{G},\mu\rangle$ and choosing the point u of $\mathcal{G}$, we have that $\langle\langle\mathcal{G},\mu\rangle,u\rangle \models \varphi$. By Theorem 13, Spoiler has a winning strategy in the r-round $\mathrm{HGL}_r(\mathsf{P},\mathsf{N})$-game on $\langle\langle\mathcal{G},\mu\rangle,u\rangle$ and $\langle\langle\mathcal{H},\lambda\rangle,v\rangle$. Hence, Spoiler has a winning strategy in the r-round colouring $\mathrm{HGL}(\mathsf{P},\mathsf{N})$-game on $\mathcal{G}$ and $\mathcal{H}$. Alternatively, if $\mathcal{G} \not\models \varphi$ and $\mathcal{H} \models \varphi$ then an analogous argument yields the same conclusion. The result follows. □

Given the relationship in Theorem 16, let us look more closely at what it means for Spoiler or Duplicator to have a winning strategy in some r-round colouring $\mathrm{HGL}(\mathsf{P},\mathsf{N})$-game on two *strongly connected* frames $\mathcal{G}$ and $\mathcal{H}$. Suppose that Spoiler has a winning strategy and that in a particular play of the game, after the colouring-start phase Spoiler's strategy is such that a $\Diamond^+$-move, a $\Box^+$-move or a $@_n$-move is made but not as the first move of the r-round $\mathrm{HGL}(\mathsf{P},\mathsf{N})$-game (on the two pointed Kripke structures). We observe that because $\mathcal{G}$ and $\mathcal{H}$ are strongly connected (and there is no 'history' associated with any play), Spoiler's strategy can be amended so that if there is a need for Spoiler to make a $\Diamond^+$-move, a $\Box^+$-move or a $@_n$-move then Spoiler necessarily makes this move as the first move of the play and thereafter only makes $\Diamond$-moves or $\Box$-moves. We emphasise that this is only because the frames are strongly connected. This observation considerably simplifies the strategies of Spoiler that Duplicator has to contend with and we implicitly assume that all Spoiler's winning strategies are of this type in what follows.

Given the above observation, we also note that we can characterize exactly when Duplicator has a winning strategy in some r-round $\mathrm{HGL}(\mathsf{P},\mathsf{N})$-game on two strongly connected pointed Kripke structures $\langle\langle\mathcal{G},\mu\rangle,u\rangle$ and $\langle\langle\mathcal{H},\lambda\rangle,v\rangle$. From above, this is self-evidently when:

- Duplicator has a winning strategy in the r-round $\mathrm{HGL}(\mathsf{P},\mathsf{N})$-game on $\langle\langle\mathcal{G},\mu\rangle,u\rangle$ and $\langle\langle\mathcal{H},\lambda\rangle,v\rangle$ but where the only moves allowed are $\Diamond$-moves and $\Box$-moves; and
- given any point x of $\langle\mathcal{G},\mu\rangle$ (resp. $\langle\mathcal{H},\lambda\rangle$) there is a point y of $\langle\mathcal{H},\lambda\rangle$ (resp. $\langle\mathcal{G},\mu\rangle$) such that Duplicator has a winning strategy in the $(r-1)$-round $\mathrm{HGL}(\mathsf{P},\mathsf{N})$-game on $\langle\langle\mathcal{G},\mu\rangle,x\rangle$ and $\langle\langle\mathcal{H},\lambda\rangle,y\rangle$ (resp. $\langle\langle\mathcal{G},\mu\rangle,y\rangle$ and $\langle\langle\mathcal{H},\lambda\rangle,x\rangle$) but where the only moves allowed are $\Diamond$-moves and $\Box$-moves.

4 Playing games in Hybrid Graph Logic

We are now in a position to prove expressibility results as regards the problems definable in various fragments of Hybrid Graph Logic.

4.1 Variable quantifier-rank

We start by focussing on hierarchies obtained by fixing the number of propositional symbols and nominals and allowing the quantifier-rank to vary.

Consider the situation where we have no propositional symbols or nominals at our disposal.

Lemma 17 Let $r \geq 0$. We have that $\mathrm{HGL}_r(\emptyset, \emptyset) \subset \mathrm{HGL}_{r+1}(\emptyset, \emptyset)$.

Proof Suppose first that $r \geq 1$ and denote the directed path of length r by $\mathcal{P}_r$; that is, the frame that as a digraph consists of a directed path of r edges. Consider the r-round colouring $\mathrm{HGL}_r(\emptyset, \emptyset)$-game on $\mathcal{P}_r$ and $\mathcal{P}_{r+1}$. Note that the colouring-start phase of any play consists of Spoiler and Duplicator each choosing just a start point in one of $\mathcal{P}_r$ and $\mathcal{P}_{r+1}$.

Suppose that Spoiler chooses the point v of $\mathcal{P}_{r+1}$ in the colouring-start phase of some play. Let l be the length of the path from v to the end-point of $\mathcal{P}_{r+1}$. Duplicator chooses the point u in $\mathcal{P}_r$ so that the length of the path from u to the end-point of $\mathcal{P}_r$ is $\min\{l, r\}$. No matter how Spoiler plays in the subsequent play (of the r-round $\mathrm{HGL}_r(\emptyset, \emptyset)$-game on $\langle \mathcal{P}_r, u \rangle$ and $\langle \mathcal{P}_{r+1}, v \rangle$), Duplicator can clearly win the play.

Alternatively, suppose that Spoiler chooses the point u of $\mathcal{P}_r$ in the colouring-start phase. Let l be the length of the path from u to the end-point of $\mathcal{P}_r$. Duplicator chooses the point v in $\mathcal{P}_{r+1}$ so that the length of the path from v to the end-point of $\mathcal{P}_{r+1}$ is l. No matter how Spoiler plays in the subsequent play, Duplicator can clearly win the play.

Hence, Duplicator has a winning strategy in the r-round colouring $\mathrm{HGL}_r(\emptyset, \emptyset)$-game on $\mathcal{P}_r$ and $\mathcal{P}_{r+1}$. By Theorem 16, $\mathcal{P}_r \equiv_{\mathrm{HGL}_r(\emptyset,\emptyset)} \mathcal{P}_{r+1}$.

Consider the formula φ_{r+1} defined as $\neg\Diamond\Diamond \ldots \Diamond\top$, where $\Diamond$ is repeated $r+1$ times. Clearly, $\mathcal{P}_r \models \varphi_{r+1}$ as no matter which point we start from, taking an $r+1$-edge walk is impossible. However, if we choose the first point of $\mathcal{P}_{r+1}$ to start from then we can take an $r+1$-edge walk; thus, $\mathcal{P}_{r+1} \not\models \varphi_{r+1}$. As φ_{r+1} has quantifier-rank $r+1$, we have that $\mathrm{HGL}_r(\emptyset, \emptyset) \subset \mathrm{HGL}_{r+1}(\emptyset, \emptyset)$.

The result follows by noting that only the problem consisting of every digraph and the problem consisting of no digraphs can be defined by formulae of $\mathrm{HGL}_0(\emptyset, \emptyset)$. $\square$

We now move on to the situation where we have no propositional symbols at our disposal but do have at least 1 nominal. For any $m \geq 1$, denote the directed cycle on m vertices by $\mathcal{C}_m$. Our basic strategy is to play games on two cycles whose lengths are 'long enough', but differ by 1, and where these lengths are at the 'cusp' in relation to the numeric parameters relating to the logic $\mathrm{HGL}_r(0, d)$. As we'll see, the fact that our cycle lengths are 'long enough' will enable us to prove equivalence in $\mathrm{HGL}_r(0, d)$, in Lemma 18, and the fact that we work at the 'cusp' means that we'll be able to tell the cycles apart in $\mathrm{HGL}_{r+1}(0, d)$, in Lemma 19.

Lemma 18 Let $r \geq 1$ and $d \geq 1$, and define $m = d(r+1)$. We have that $\mathcal{C}_{m+1} \equiv_{\mathrm{HGL}_r(0,d)} \mathcal{C}_{m+2}$.

Proof Let $\mathsf{N} = \{n_0, n_1, \ldots, n_{d-1}\}$ be a set of $d \geq 1$ nominals. For brevity, denote $\mathcal{C}_{m+1}$ by $\mathcal{G} = \langle V^{\mathcal{G}}, E^{\mathcal{G}} \rangle$ and $\mathcal{C}_{m+2}$ by $\mathcal{H} = \langle V^{\mathcal{H}}, E^{\mathcal{H}} \rangle$. Consider the r-round colouring $\mathrm{HGL}(\emptyset, \mathsf{N})$-game on $\mathcal{G}$ and $\mathcal{H}$.

Suppose that in the colouring-start phase Spoiler chooses the valuation function $\mu : \mathsf{N} \to V^{\mathcal{G}}$ and the point u. Suppose that in the first instance $\mu(n_0) = \mu(n_1) = \ldots = \mu(n_{d-1})$. Duplicator chooses the valuation function $\lambda : \mathsf{N} \to V^{\mathcal{H}}$ so that $\lambda(n_0) = \lambda(n_1) = \ldots = \lambda(n_{d-1})$ and chooses v so that the length of the path from v to $\lambda(n_0)$ in $\mathcal{H}$ is equal to the length of the path from u to $\mu(n_0)$ in $\mathcal{G}$. Duplicator clearly has a winning strategy in the subsequent r-round $\mathrm{HGL}(\emptyset, \mathsf{N})$-game on $\langle\langle \mathcal{G}, \mu\rangle, u\rangle$ and $\langle\langle \mathcal{H}, \lambda\rangle, v\rangle$. The only remark to make is that if Spoiler's first move is a $\Box^+$-move so that b_1 is such that there is an edge from $\lambda(n_0)$ to b_1 (that is, the path from b_1 to $\lambda(n_0)$ has length $m+1$) then Duplicator replies so that there is an edge from $\mu(n_0)$ to a_1 (that is, the path from a_1 to $\mu(n_0)$ has length m). Duplicator can always successfully respond to Spoiler's subsequent $r-1$ $\Diamond$- or $\Box$-moves.

Alternatively, suppose that in the colouring-start phase Spoiler begins by choosing the valuation function $\lambda : \mathsf{N} \to V^{\mathcal{H}}$ and the point v. Similarly to as above, suppose that $\lambda(n_0) = \lambda(n_1) = \ldots = \lambda(n_{d-1})$. Let the length of the path from v to $\lambda(n_0)$ be l. Duplicator chooses the valuation function $\mu : \mathsf{N} \to V^{\mathcal{G}}$ so that $\mu(n_0) = \mu(n_1) = \ldots = \mu(n_{d-1})$, and she chooses u so that the length of the path from u to $\mu(n_0)$ in $\mathcal{G}$ is $\min\{l, m\}$ (note that l might be equal to $m+1$). Clearly (as in the previous paragraph), Duplicator has a winning strategy in the subsequent r-round HGL($\emptyset$, N)-game on $\langle\langle\mathcal{G},\mu\rangle,u\rangle$ and $\langle\langle\mathcal{H},\lambda\rangle,v\rangle$.

Now suppose that in the colouring-start phase Spoiler chooses the valuation function $\mu : \mathsf{N} \to V^{\mathcal{G}}$ and the point u where w.l.o.g.: $\lambda(n_0) \neq \lambda(n_1)$ (and so $d \geq 2$); the nominals appear on the cycle $\mathcal{G}$ in the order $n_0, n_1, \ldots, n_{d-1}$ (consecutive nominals might sit on the same vertex); and u lies on the path from $\lambda(n_0)$ to $\lambda(n_1)$ in $\mathcal{G}$ but where $u \neq \lambda(n_1)$ (it might be the case that $u = \lambda(n_0)$, though). Duplicator replies by choosing the valuation function $\lambda : \mathsf{N} \to V^{\mathcal{H}}$ and point v as follows. Duplicator takes a copy of $\langle\langle\mathcal{G},\mu\rangle,u\rangle$ and 'splits' an edge by replacing it with a path of length 2, with the edge to be split chosen as we now describe:

1. if the length of the path from $\mu(n_0)$ to $\mu(n_1)$ is at most $r+2$ then there must be some n_i, where $i \neq 0$, so that the length of the path from $\mu(n_i)$ to $\mu(n_{i+1})$ (with addition modulo d) is at least $r+2$; choose the first edge on this path to split

2. if the length of the path from $\mu(n_0)$ to $\mu(n_1)$ is at least $r+3$ and $v = \mu(n_0)$ then split the first edge of this path

3. if the length of the path from $\mu(n_0)$ to $\mu(n_1)$ is at least $r+3$ and $v \neq \mu(n_0)$ then split the first edge of the path from $\mu(n_0)$ to v.

The pointed Kripke structure $\langle\langle\mathcal{H},\lambda\rangle,v\rangle$ is obtained from this 'split-edge' structure by setting $\lambda(n_i) = \mu(n_i)$, for $i \in \{0,1,\ldots,d-1\}$, and renaming u to v. The three different constructions can be visualized in Fig. 1 (where the grey vertex is the 'new' vertex and the white vertex is u or v).

It is not difficult to see that Duplicator has a winning strategy in the subsequent r-round HGL($\emptyset$, N)-game on $\langle\langle\mathcal{G},\mu\rangle,u\rangle$ and $\langle\langle\mathcal{H},\lambda\rangle,v\rangle$. The only remark to make is that if Spoiler's first move is a $\Box^+$-move so that b_1 is the 'new' point, in $\mathcal{H}$, introduced by splitting the edge (x,y) of $\mathcal{G}$ then a_1 is chosen to be the point y of $\mathcal{G}$.

Alternatively, suppose that Spoiler begins the colouring-start phase by choosing the valuation function $\lambda : \mathsf{N} \to V^{\mathcal{H}}$ and the point v where w.l.o.g.: $\lambda(n_0) \neq \lambda(n_1)$ (and so $d \geq 2$); the nominals appear on the cycle $\mathcal{H}$ in the order $n_0, n_1, \ldots, n_{d-1}$ (consecutive nominals might sit on the same vertex); and v lies on the path from $\lambda(n_0)$ to $\lambda(n_1)$ in $\mathcal{H}$ but where $v \neq \lambda(n_1)$ (it might be the case that $v = \lambda(n_0)$, though). Duplicator chooses the valuation function $\mu : \mathsf{N} \to V^{\mathcal{G}}$ and point u as follows. Duplicator takes a copy of $\langle\langle\mathcal{H},\lambda\rangle,v\rangle$ and 'merges' a path of length 2 by replacing it with an edge, with the path to be merged chosen as we now describe:

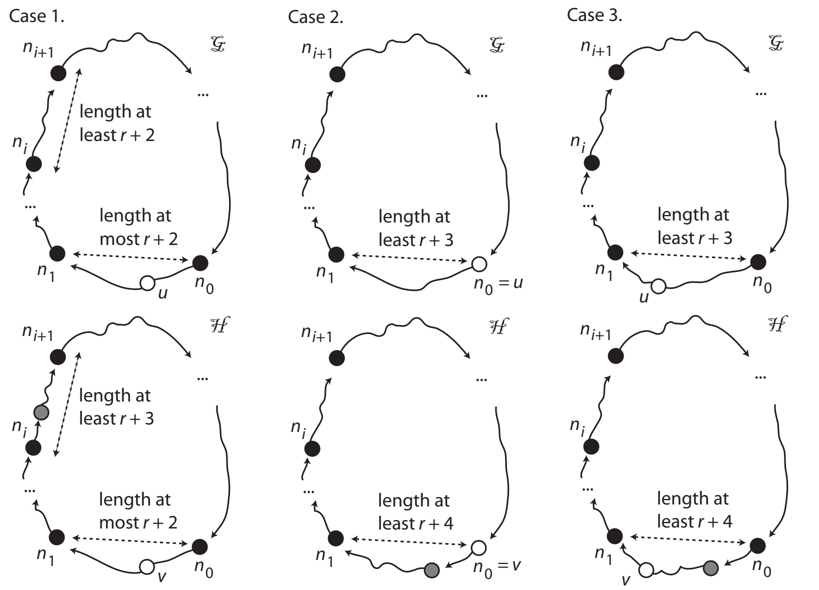

Figure 1. Building $\langle\langle \mathcal{H}, \lambda\rangle, v\rangle$ from $\langle\langle \mathcal{G}, \mu\rangle, u\rangle$.

1. if the length of the path from $\lambda(n_0)$ to $\lambda(n_1)$ is at most $r + 2$ then there must be some n_i, where $i \neq 0$, so that the length of the path from $\lambda(n_i)$ to $\lambda(n_{i+1})$ (with addition modulo d) is at least $r + 2$; choose the first 2 edges of this path to merge (so that the point common to both edges is removed)

2. if the length of the path from $\lambda(n_0)$ to $\lambda(n_1)$ is at least $r + 3$ and either $v = \lambda(n_0)$ or there is an edge from $\lambda(n_0)$ to v then choose the last 2 edges of this path to merge

3. if the length of the path from $\lambda(n_0)$ to $\lambda(n_1)$ is at least $r + 3$ and $v \neq \lambda(n_0)$ and there is no edge from $\lambda(n_0)$ to v then choose the first 2 edges of this path to merge.

The pointed Kripke structure $\langle\langle \mathcal{G}, \mu\rangle, u\rangle$ is obtained from this 'merge-edge' structure by setting $\mu(n_i) = \lambda(n_i)$, for $i \in \{0, 1, \ldots, d-1\}$, and renaming v to u. The different constructions can be visualized in Fig. 2.

Again, it is not difficult to see that Duplicator has a winning strategy in the subsequent r-round HGL($\emptyset$, N)-game on $\langle\langle \mathcal{G}, \mu\rangle, u\rangle$ and $\langle\langle \mathcal{H}, \lambda\rangle, v\rangle$. Consequently, the result follows from Theorem 16. □

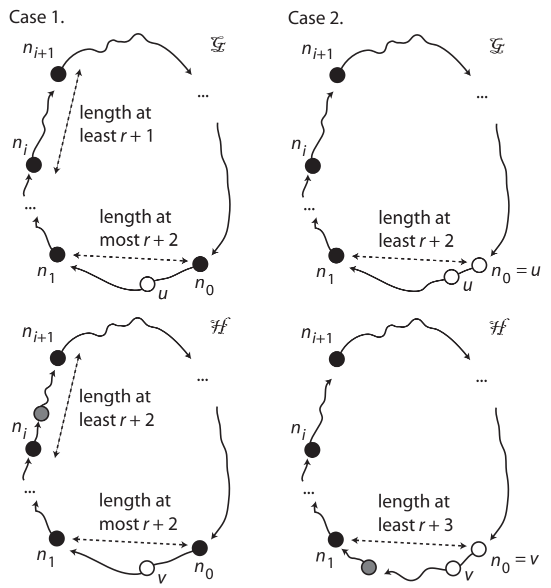

Figure 2. Building $\langle\langle\mathcal{G},\mu\rangle,u\rangle$ from $\langle\langle\mathcal{H},\lambda\rangle,v\rangle$.

Lemma 19 Let $r \geq 1$ and $d \geq 1$, and define $m = d(r+1)$. There exists a formula φ of $\mathrm{HGL}_{r+1}(0,d)$ such that $C_{m+1} \models \varphi$ and $\mathcal{C}_{m+2} \not\models \varphi$.

Proof Let $\mathsf{N} = \{n_0, n_1, \ldots, n_{d-1}\}$ be a set of $d \geq 1$ nominals. For brevity, denote $\mathcal{C}_{m+1}$ by $\mathcal{G} = \langle V^{\mathcal{G}}, E^{\mathcal{G}}\rangle$ and $\mathcal{C}_{m+2}$ by $\mathcal{H} = \langle V^{\mathcal{H}}, E^{\mathcal{H}}\rangle$. We shall prove that Spoiler has a winning strategy in the $(r+1)$-round colouring $\mathrm{HGL}(\emptyset,\mathsf{N})$-game on $\mathcal{G}$ and $\mathcal{H}$. The result then follows by Theorem 16.

Consider a play in the $(r+1)$-round colouring $\mathrm{HGL}(\emptyset,\mathsf{N})$-game on $\mathcal{G}$ and $\mathcal{H}$. Suppose that $d > 1$. In Spoiler's colouring-start move: Spoiler chooses $\lambda : \mathsf{N} \to V^{\mathcal{H}}$ such that the nominals $n_0, n_1, \ldots, n_{d-1}$ appear in this order as we traverse the directed cycle $\mathcal{H}$ and so that: the length of the path from $\lambda(n_0)$ to $\lambda(n_1)$ is $r+3$; for $i = 1, 2, \ldots, d-2$, the length of the path from the point $\lambda(n_i)$ to the point $\lambda(n_{i+1})$ is $r+1$; and the length of the path from the point $\lambda(n_{d-1})$ to the point $\lambda(n_0)$ is $r+1$. In addition, Spoiler chooses v as the point x for which there is an edge $(\lambda(n_0), x)$.

Consider Duplicator's response if she wishes to progress to a win. No matter which valuation function μ and point u Duplicator chooses, the path of length $r+1$ starting at u in $\langle\langle\mathcal{G},\mu\rangle,u\rangle$ must be nominal-free (otherwise, Spoiler would walk along the path starting at v in $\mathcal{H}$ via $r+1$ $\Box$-moves). Hence, there are two nominals, n_i and n_j, say, so that there is a nominal-free path from $\mu(n_i)$ to $\mu(n_j)$ in $\langle\langle\mathcal{G},\mu\rangle,u\rangle$ of length at least $r+3$. Consequently, there are two nominals, $n_{i'}$ and $n_{j'}$, say, so that there is a nominal-free path from $\mu(n_{i'})$ to $\mu(n_{j'})$ in $\mathcal{G}$ of length at at most r (recall, $\mathcal{G}$ is a cycle of length $d(r+1)+1$ and $d \geq 2$). Spoiler now makes a $@_{n_{i'}}$-move so that $a_1 = \mu(n_{i'})$, with Duplicator necessarily having to ensure that $b_1 = \lambda(n_{i'})$. Spoiler now makes r $\Diamond$-moves which results in him winning the play.

Suppose that $d = 1$. Spoiler proceeds as follows. After choosing $\lambda(n_0)$, Spoiler chooses v as the point x where there is an edge $(\lambda(n_0), x)$. No matter how Duplicator replies, when Spoiler subsequently makes $r+1$ $\Diamond$-moves, Spoiler wins the play. Hence, Spoiler has a winning strategy in the $(r+1)$-round colouring HGL$(\emptyset, \mathsf{N})$-game on $\mathcal{G}$ and $\mathcal{H}$. The result follows. □

Our proof of Lemma 19 is via establishing a winning strategy for Spoiler. However, we can also obtain this result by presenting an explicit formula of $\mathrm{HGL}_{r+1}(0,d)$ that tells C_{m+1} and $\mathcal{C}_{m+2}$ apart. It is quicker to proceed as we have done above and also instructive to fully appreciate the power of game playing. Nevertheless, let us state here an explicit formula of $\mathrm{HGL}_{r+1}(0,d)$ that does indeed tell $\mathcal{C}_{m+1}$ and $\mathcal{C}_{m+2}$ apart (we leave it up to the reader to verify that this is the case).

- Define χ as $\neg n_0 \wedge \neg n_1 \wedge \ldots \wedge \neg n_{d-1}$. So, χ holds at some point of some Kripke structure if no nominal sits on this point.

- Define η as $\bigwedge_{i \neq j}(\neg n_i \vee \neg n_j)$, where $i, j \in \{0, 1, \ldots, d-1\}$. So, η holds at some point of some Kripke structure if at most one nominal sits on this point.

- Define φ_0 as χ and for $i \geq 1$, define φ_i as $\chi \wedge \Diamond(\varphi_{i-1})$. So, φ_i holds at some point of some Kripke structure if there is a path of length i from this point so that no nominal sits on any point of this path.

- Define ψ_1 as $\Diamond\chi$ and for $i \geq 2$, define ψ_i as $\Diamond(\chi \wedge \psi_{i-1})$. So, ψ_i holds at some point of some Kripke structure if there is a path of length i from this point so that no nominal sits on any point of this path apart from possibly the first.

- If $d = 1$, $r \geq 1$ and $m = r+1$ then we have that $\mathcal{C}_{m+1} \models \neg\varphi_{r+1}$ but $\mathcal{C}_{m+2} \not\models \neg\varphi_{r+1}$.

- Define Φ as $\eta \wedge \varphi_{r+1} \wedge \bigwedge_{i=0}^{d-1} @_{n_i}\psi_r$. If $d \geq 2$, $r \geq 1$ and $m = d(r+1)$ then we have that $\mathcal{C}_{m+1} \models \neg\Phi$ but $\mathcal{C}_{m+2} \not\models \neg\Phi$.

Note that we have not used $\Diamond^+$ and $\Box^+$ in the construction of the above formula.

The following lemma is immediate.

Lemma 20 Let $c \geq 0$ and $d \geq 0$. Let $\mathcal{G}$ be the frame with one point and no edges, and let $\mathcal{H}$ be the frame with one point and a self-loop. We have that $\mathcal{G} \equiv_{\mathrm{HGL}_0(c,d)} \mathcal{H}$ but that $\mathcal{G} \models \neg\Diamond\top$ and $\mathcal{H} \not\models \neg\Diamond\top$.

Lemmas 18, 19 and 20 yield the following result.

Theorem 21 When $r \geq 0$ and $d \geq 1$, we have that

$$\mathrm{HGL}_r(0,d) \subset \mathrm{HGL}_{r+1}(0,d).$$

We now turn to when we have access to propositional symbols as well as nominals. We denote a path ρ in some digraph from vertex s to vertex t by $\rho(s,t)$. For $i \geq 1$ and $j \geq 0$, define the digraph $\mathcal{G}_{i,j}$ as follows:

- take the disjoint union of i directed paths of length j, namely the paths $\rho_1(s_1,t_1), \rho_2(s_2,t_2), \ldots, \rho_i(s_i,t_i)$ (where if $j = 0$ then $s_1 = t_1, s_2 = t_2, \ldots, s_i = t_i$)

- include also the distinct vertex s along with the edges of $\{(s,s_l),(t_l, s) : l = 1,2,\ldots,i\}$.

The digraph $\mathcal{G}_{i,j}$ can be visualized as in Fig. 3.

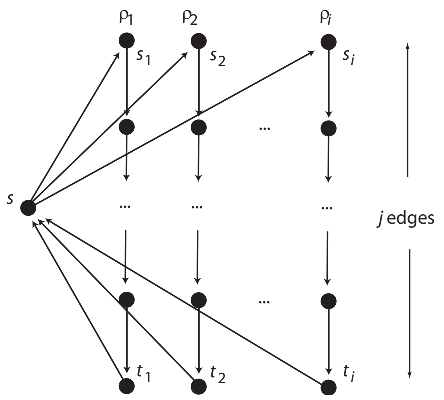

Figure 3. The digraph $\mathcal{G}_{i,j}$.

Lemma 22 Let $r \geq 1$, $c \geq 1$ and $d \geq 0$. Define $m = d + 2^{c(r+1)}$. We have that $\mathcal{G}_{m-1,r} \equiv_{\mathrm{HGL}_r(c,d)} \mathcal{G}_{m,r}$.

Proof For brevity, denote $\mathcal{G}_{m-1,r}$ by $\mathcal{G} = \langle V^{\mathcal{G}}, E^{\mathcal{G}} \rangle$ and denote $\mathcal{G}_{m,r}$ by $\mathcal{H} = \langle V^{\mathcal{H}}, E^{\mathcal{H}} \rangle$. A path ρ_i in $\mathcal{G}$ or $\mathcal{H}$ will be denoted $\rho_i^{\mathcal{G}}$ or $\rho_i^{\mathcal{H}}$, depending upon whether it lies in $\mathcal{G}$ or $\mathcal{H}$, respectively, and the same goes for vertices (so, we have the vertex $s^{\mathcal{G}}$ of $V^{\mathcal{G}}$, the vertex $s^{\mathcal{H}}$ of $V^{\mathcal{H}}$, and so on).

Let P be a set of c propositional symbols and let N be a set of d nominals. Consider a play of the r-round colouring HGL(P, N)-game on $\mathcal{G}$ and $\mathcal{H}$. Suppose that Spoiler makes a colouring-start move so as to build the pointed Kripke $\mathsf{P} \cup \mathsf{N}$-structure $\langle \langle \mathcal{H}, \lambda \rangle, v \rangle$. This results in each path $\rho_i^{\mathcal{H}}$ having a specific *colour-type*: an ordered list of $r + 1$ subsets of $\mathsf{P} \cup \mathsf{N}$, one for each vertex w of $\rho_i^{\mathcal{H}}$, detailing for which $p \in \mathsf{P}$ or $n \in \mathsf{N}$ we have that $w \in \lambda(p)$ or $w \in \lambda(n)$. We call a path $\rho_i^{\mathcal{H}}$ *clean* if no nominal is involved in any of the $r + 1$ sets of the colour-type of $\rho_i^{\mathcal{H}}$, and *dirty* otherwise; we extend the definition of clean and dirty to colour-types also. W.l.o.g. we may assume that none of the paths of $\{\rho_{d+1}^{\mathcal{H}}, \rho_{d+2}^{\mathcal{H}}, \ldots, \rho_m^{\mathcal{H}}\}$ are dirty. We can also make the following assumptions.

1. If two clean paths from $\{\rho_1^{\mathcal{H}}, \rho_2^{\mathcal{H}}, \ldots, \rho_m^{\mathcal{H}}\}$ have the same colour-type then the path $\rho_m^{\mathcal{H}}$ has the same colour-type as some clean path from $\{\rho_1^{\mathcal{H}}, \rho_2^{\mathcal{H}}, \ldots, \rho_{m-1}^{\mathcal{H}}\}$.
2. If no two clean paths have the same colour-type then the path $\rho_m^{\mathcal{H}}$ is the path of colour-type $S, S, \ldots, S$, where $S = \{p \in \mathsf{P} : s^{\mathcal{H}} \in \lambda(p)\}$ (note that in this case: every path of $\{\rho_i^{\mathcal{H}} : i = 1, 2, \ldots, d\}$ must be dirty; every path of $\{\rho_i^{\mathcal{H}} : i = d + 1, d + 2, \ldots, m\}$ must be clean; and $s \neq \mu(n)$, for all $n \in \mathsf{N}$).

Consider $\mathcal{H}$ with the vertices of the path $\rho_m^{\mathcal{H}}$ removed (along with any incident edges); that is, $\mathcal{H} \setminus \{\rho_m^{\mathcal{H}}\}$. Let $\lambda' : \mathsf{P} \cup \mathsf{N} \to V^{\mathcal{H}} \setminus \{w : w \text{ is a vertex of } \rho_m^{\mathcal{H}}\}$ be defined via $\lambda'(p) = \lambda(p) \setminus \{w : w \text{ is a vertex of } \rho_m^{\mathcal{H}}\}$, for $p \in \mathsf{P}$, and $\lambda'(n) = \lambda(n)$, for $n \in \mathsf{N}$. For simplicity, denote the Kripke $\mathsf{P} \cup \mathsf{N}$-structure $\langle \mathcal{H} \setminus \{\rho_m^{\mathcal{H}}\}, \lambda' \rangle$ by $\langle \mathcal{H} \setminus \{\rho_m^{\mathcal{H}}\}, \lambda \rangle$. Duplicator replies with a colouring-start move so as to build the Kripke $\mathsf{P} \cup \mathsf{N}$-structure $\langle \mathcal{G}, \mu \rangle$ such that $\langle \mathcal{G}, \mu \rangle$ is isomorphic to $\langle \mathcal{H} \setminus \{\rho_m^{\mathcal{H}}\}, \lambda \rangle$, via the natural isomorphism $f : \mathcal{G} \to \mathcal{H} \setminus \{\rho_m^{\mathcal{H}}\}$. However, some care needs to be taken by Duplicator in the choice of u. If Spoiler has chosen v in the isomorphic copy of $\mathcal{G}$ in $\mathcal{H}$ (w.r.t. to the natural isomorphism f) then Duplicator chooses u in $\mathcal{G}$ according to the isomorphism f. There are other possibilities.

1. Suppose that the path $\rho_m^{\mathcal{H}}$ and the path $\rho_j^{\mathcal{H}}$, where $1 \leq j < m$, have the same (clean) colour-type. If Spoiler has chosen v to be the ith vertex of $\rho_m^{\mathcal{H}}$ then w.l.o.g. we may assume that Spoiler has chosen v as the ith vertex of $\rho_j^{\mathcal{H}}$ and Duplicator chooses u according to the natural isomorphism f.

2. Suppose that all clean paths from $\{\rho_i^{\mathcal{H}} : 1 \leq i \leq m\}$ have a unique colour-type, and so all paths of $\{\rho_i^{\mathcal{H}} : 1 \leq i \leq d\}$ must be dirty with all paths of $\{\rho_i^{\mathcal{H}} : d+1 \leq i \leq m\}$ clean. Recall that the colour-type of $\rho_m^{\mathcal{H}}$ is $S, S, \ldots, S$, where $S = \{p \in \mathsf{P} : s^{\mathcal{H}} \in \lambda(p)\}$, and that this colour-type does not appear as the colour-type of any path of $\{\rho_i^{\mathcal{G}} : 1 \leq i \leq m-1\}$. W.l.o.g. let $\rho_{d+1}^{\mathcal{H}}$ be a path whose colour-type is $S', S, \ldots, S$, for some $S' \neq S$, and let $\rho_{d+2}^{\mathcal{H}}$ be a path whose colour-type is $S, S, \ldots, S, S''$, for some $S'' \neq S$. The situation can be visualized as in Fig. 4 where a vertex w (of $V^{\mathcal{G}}$ or $V^{\mathcal{H}}$) with the property that $\{p \in \mathsf{P} : w \in \lambda(p)\} = S$ is depicted in white.

   - If Spoiler has chosen the first vertex of the path $\rho_m^{\mathcal{H}}$ as v then Duplicator chooses the second vertex of $\rho_{d+1}^{\mathcal{G}}$ as u.
   - If Spoiler has chosen the ith vertex of the path $\rho_m^{\mathcal{H}}$ as v, where $2 \leq i \leq r+1$, then Duplicator chooses the ith vertex of $\rho_{d+1}^{\mathcal{G}}$ as u.

Following the colouring-start move, there are two essential situations.

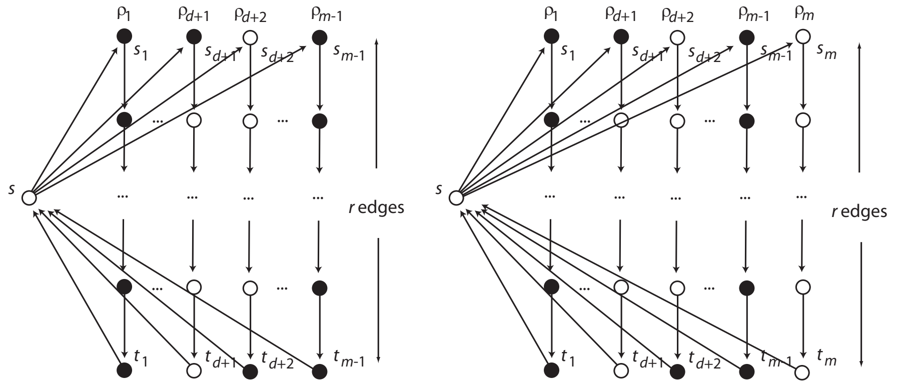

Figure 4. $\langle \mathcal{G}, \mu \rangle$ and $\langle \mathcal{H}, \lambda \rangle$ when $\langle \mathcal{H}, \lambda \rangle$ has distinct clean colour-types.

Case 1: Suppose that the colour-type of the (clean) path $\rho_m^{\mathcal{H}}$ is the same as the colour-type of the path $\rho_j^{\mathcal{H}}$, for some path $\rho_j^{\mathcal{H}}$ where $1 \leq j < m$.

Extend the natural isomorphism f^{-1} from $\mathcal{H} \setminus \{\rho_m^{\mathcal{H}}\}$ to $\mathcal{G}$ to the map g from $\mathcal{H}$ to $\mathcal{G}$ so that if w is some point of $\rho_m^{\mathcal{H}}$ then $g(w) = f^{-1}(w')$, where w' is the point of $\rho_j^{\mathcal{H}}$ analogous to w (that is, if w is the ith vertex of $\rho_m^{\mathcal{H}}$ then w' is the ith vertex of $\rho_j^{\mathcal{H}}$). Duplicator's strategy in the subsequent r-round HGL(P,N)-game on $\langle\langle\mathcal{G},\mu\rangle,u\rangle$ and $\langle\langle\mathcal{H},\lambda\rangle,v\rangle$ is simply to play according to g, if Spoiler plays in $\mathcal{H}$, or according to f, if Spoiler plays in $\mathcal{G}$. Clearly, this yields a winning strategy for Duplicator in the r-round colouring HGL(P,N)-game on $\mathcal{G}$ and $\mathcal{H}$.

Case 2: Suppose that all of the colour-types of the clean paths of $\{\rho_i^{\mathcal{H}} : 1 \leq i \leq m = d + 2^{c(r+1)}\}$ are distinct. As remarked earlier, we must have that every path of $\{\rho_1^{\mathcal{H}}, \rho_2^{\mathcal{H}}, \ldots, \rho_d^{\mathcal{H}}\}$ is dirty with every path of $\{\rho_{d+1}^{\mathcal{H}}, \rho_{d+2}^{\mathcal{H}}, \ldots, \rho_m^{\mathcal{H}}\}$ clean. By construction: $\rho_m^{\mathcal{H}}$ has colour-type $S, S, \ldots, S$; $\rho_{d+1}^{\mathcal{H}}$ has colour-type $S', S, \ldots, S$; and $\rho_{d+1}^{\mathcal{H}}$ has colour-type $S, S, \ldots, S, S''$ (with S, S' and S'' defined as above).

Consider Spoiler's moves in a play of the r-round HGL(P,N)-game on $\langle\langle\mathcal{G},\mu\rangle,u\rangle$ and $\langle\langle\mathcal{H},\lambda\rangle,v\rangle$. Recall that we may assume that if Spoiler makes a $\Diamond^+$-move, a $\Box^+$-move or a $@_n$-move then this is will be the first move of the play and every subsequent move will be a $\Diamond$-move or a $\Box$-move (indeed, we may assume that no $@_n$-move is ever made by Spoiler as Spoiler can incorporate such a move into the choice of v). Note also that Duplicator's reply to a move of Spoiler involves no choice unless either: the pebble Duplicator is moving happens to lie on $s^{\mathcal{G}}$ or $s^{\mathcal{H}}$; or Spoiler's move is a $\Diamond^+$-move or a $\Box^+$-move, either of which can only happen if the move is the first move of the play. If Spoiler's (first) move is a $\Diamond^+$-move then Duplicator simply plays according to the natural isomorphism f.

The strategy for Duplicator will be such as to force that whenever pebble a is on $s^{\mathcal{G}}$, we have that pebble b is on $s^{\mathcal{H}}$, and *vice versa*, apart from the possibility that after the rth move of the play pebble a is on $s^{\mathcal{G}}$ and pebble b is not on $s^{\mathcal{H}}$. If it is the case that pebble a is on $s^{\mathcal{G}}$ and pebble b is on $s^{\mathcal{H}}$, and less than r moves have been made, then unless Spoiler makes a $\Box$-move or a $\Box^+$-move to a vertex of $\rho_m^{\mathcal{H}}$, Duplicator plays according to the natural isomorphism f. If Spoiler makes a $\Box$-move and places pebble b on $s_m^{\mathcal{H}}$ then Duplicator replies by placing pebble a on $s_{d+2}^{\mathcal{G}}$.

Suppose that Spoiler makes a $\Box^+$-move (as the first move and irrespective of the choice of v). If Spoiler places the pebble b on a point of $\mathcal{H} \setminus \{\rho_m^{\mathcal{H}}\}$ then Duplicator places pebble a according to the natural isomorphism. If Spoiler places the pebble b on a point of $\rho_m^{\mathcal{H}}$ then Duplicator's reply is as follows: if Spoiler plays on the first vertex of $\rho_m^{\mathcal{H}}$ then

Duplicator replies by playing on the second vertex of $\rho^{\mathcal{G}}_{d+1}$; and if Spoiler plays on the ith vertex of $\rho^{\mathcal{H}}_m$, where $2 \leq i \leq r+1$, then Duplicator replies by playing on the ith vertex of $\rho^{\mathcal{G}}_{d+1}$. Note that because all of the subsequent $r-1$ moves are $\Diamond$-moves or $\Box$-moves, Duplicator can clearly win this play of the game:

- if b_1 is the first vertex of $\rho^{\mathcal{H}}_m$ and a_1 is the second vertex of $\rho^{\mathcal{G}}_{d+2}$ then in the remainder of the play the pebbles never leave the paths $\rho^{\mathcal{H}}_m$ and $\rho^{\mathcal{G}}_{d+2}$;
- if b_1 is the ith vertex of $\rho^{\mathcal{H}}_m$ and a_1 is the ith vertex of $\rho^{\mathcal{G}}_{d+1}$, for some $i \in \{2, 3, \ldots, r+1\}$, then by playing as directed above (when the pebbles arrive at $s^{\mathcal{G}}$ and $s^{\mathcal{H}}$), Duplicator can win the play.

Finally, we note that when u and v are as defined above, Duplicator can win the play by playing as directed above. The only slightly awkward situation is when u is the second vertex of $\rho^{\mathcal{G}}_{d+1}$, v is the first vertex of $\rho^{\mathcal{H}}_m$ and all r moves are $\Diamond$- or $\Box$-moves. However, the fact that the path in $\mathcal{G}$ consisting of the last r vertices of $\rho^{\mathcal{G}}_{d+1}$ augmented with the vertex $s^{\mathcal{G}}$ has the same colour-type as the path $\rho^{\mathcal{H}}_m$ enables Duplicator to win the play.

Consequently, Duplicator has a winning strategy in the r-round colouring HGL(P, N)-game on $\mathcal{G}$ and $\mathcal{H}$.

Alternatively, suppose that Spoiler's colouring-start move is so as to build the pointed Kripke $\mathsf{P} \cup \mathsf{N}$-structure $\langle\langle\mathcal{G}, \mu\rangle, u\rangle$. W.l.o.g. we may assume that the path $\rho^{\mathcal{G}}_{m-1}$ is clean. Duplicator builds the Kripke structure $\mathsf{P} \cup \mathsf{N}$-structure $\langle\mathcal{H}, \lambda\rangle$ by taking a copy of $\langle\mathcal{G}, \mu\rangle$ and extending it with the path $\rho^{\mathcal{H}}_m$ and edges from (resp. to) $s^{\mathcal{H}}$ to the first (resp. from the last) vertex of $\rho^{\mathcal{H}}_m$ so that the colour-type of $\rho^{\mathcal{H}}_m$ is identical to the colour-type of $\rho^{\mathcal{G}}_{m-1}$. There is a natural embedding of $\langle\mathcal{G}, \mu\rangle$ in $\langle\mathcal{H}, \lambda\rangle$, and the point v is taken to be the image of u under this embedding; this is the pointed Kripke structure $\langle\langle\mathcal{H}, \lambda\rangle, v\rangle$. Given our more complicated arguments above, it should be clear that Duplicator has a winning strategy in the r-round colouring HGL(P, N)-game on $\mathcal{G}$ and $\mathcal{H}$. The result follows by Theorem 16. $\Box$

Lemma 23 Let $r \geq 1$, $c \geq 1$ and $d \geq 0$. Define $m = d + 2^{c(r+1)}$. There exists a formula φ of HGL$_{r+1}(c, d)$ such that $\mathcal{G}_{m-1,r} \models \varphi$ and $\mathcal{G}_{m,r} \not\models \varphi$.

Proof Let P be a set of c propositional symbols and let $\mathsf{N} = \{n_0, n_1, \ldots, n_{d-1}\}$ be a set of d nominals. For brevity, denote $\mathcal{G}_{m-1,r}$ by $\mathcal{G} = \langle V^{\mathcal{G}}, E^{\mathcal{G}}\rangle$ and $\mathcal{G}_{m,r}$ by $\mathcal{H} = \langle V^{\mathcal{H}}, E^{\mathcal{H}}\rangle$.

Consider the following strategy by Spoiler in the $(r+1)$-move colouring HGL(P,N)-game on $\mathcal{G}$ and $\mathcal{H}$. Spoiler begins by choosing the valuation function $\lambda : \mathsf{P} \cup \mathsf{N} \to V^{\mathcal{H}}$ so that if $d \geq 1$ then the paths of $\{\rho_1^{\mathcal{H}}, \rho_2^{\mathcal{H}}, \ldots, \rho_d^{\mathcal{H}}\}$ are all dirty, and so that the colour-types of the paths of $\{\rho_{d+1}^{\mathcal{H}}, \rho_{d+2}^{\mathcal{H}}, \ldots, \rho_m^{\mathcal{H}}\}$ are all distinct. Moreover, if $d \geq 1$ then Spoiler chooses λ so that $\lambda(n_i)$ is the first vertex of $\rho_{i+1}^{\mathcal{H}}$, for $i = 0, 1, \ldots, d-1$. Spoiler chooses v as $s^{\mathcal{H}}$. Duplicator replies with some valuation function $\mu : \mathsf{P} \cup \mathsf{N} \to V^{\mathcal{G}}$ and in order to stand a chance of winning the play, Duplicator must clearly choose u as $s^{\mathcal{G}}$. If $d \geq 1$ then in order for Duplicator to stand a chance of winning the play, we must clearly have that w.l.o.g. $\mu(n_i)$ is the first vertex of $\rho_{i+1}^{\mathcal{G}}$, for $i = 0, 1, \ldots, d-1$.

Consequently, the paths of $\{\rho_1^{\mathcal{G}}, \rho_2^{\mathcal{G}}, \ldots, \rho_d^{\mathcal{G}}\}$ are all dirty and the paths of $\{\rho_{d+1}^{\mathcal{G}}, \rho_{d+2}^{\mathcal{G}}, \ldots, \rho_{m-1}^{\mathcal{G}}\}$ are all clean. There must be some path $\rho_j^{\mathcal{H}}$ in $\{\rho_{d+1}^{\mathcal{H}}, \rho_{d+2}^{\mathcal{H}}, \ldots, \rho_m^{\mathcal{H}}\}$ whose colour-type does not appear amongst the colour-types of the paths of $\{\rho_{d+1}^{\mathcal{G}}, \rho_{d+2}^{\mathcal{G}}, \ldots, \rho_{m-1}^{\mathcal{G}}\}$. Spoiler next makes $r+1$ $\Box$-moves and walks along the path $\rho_j^{\mathcal{H}}$. Clearly, this results in a winning play for Spoiler. Hence, Spoiler has a winning strategy for the $(r+1)$-move colouring HGL(P,N)-game on $\mathcal{G}$ and $\mathcal{H}$. The result follows by Theorem 16. $\Box$

Just as was the case with Lemma 19, we can prove Lemma 23 by constructing an explicit formula of $\text{HGL}_{r+1}(c,d)$ to tell $\mathcal{G}_{m-1,r}$ and $\mathcal{G}_{m,r}$ apart. This time, it is much more complicated and proceeding by using our game makes life much easier. However, let us us describe such a formula. Essentially, given some pointed Kripke structure $\langle\langle\mathcal{G},\mu\rangle,u\rangle$, our formula will be the negation of the formula Φ that says: 'from u, upon which no nominal sits, we can always move along an edge so that thereafter there is a path of length r upon which no nominal sits and whose colour-type is any colour-type involving just the propositional symbols from the set P; moreover, from u, we can move to a vertex on which exactly one nominal sits and where this nominal can be any nominal of N'. This formula is easy to construct. For example, suppose that $\mathsf{P} = \{p_1, p_2\}$ and $\mathsf{N} = \{n\}$. To say that we can move along an edge to a point where no nominal sits and where thereafter there is a path of length 2 every point of which has colour-type $\{p_1\}$ and upon which no nominal sits, we can use the formula $\Diamond(\neg n \wedge \Diamond(p_1 \wedge \neg p_2 \wedge \neg n \wedge \Diamond(p_1 \wedge \neg p_2 \wedge \neg n)))$. We note that it is possible to construct this formula so that it is in $HGL_{r+1}(c,d)$ and contains no applications of $\Diamond^+$ or $\Box^+$. From above, it is not difficult to see that $\mathcal{G}_{m-1,r} \models \neg\Phi$ whereas $\mathcal{G}_{m,r} \not\models \neg\Phi$.

The following result is immediate from Theorem 21 and Lemmas 20, 22 and 23.

Theorem 24 When $r \geq 0$, $c \geq 0$ and $d \geq 0$, we have

$$\mathrm{HGL}_r(c,d) \subset \mathrm{HGL}_{r+1}(c,d).$$

Indeed, taking into account our remarks after the proofs of Lemmas 19 and 23, we can say more.

Theorem 25 Suppose that $r \geq 0$, $c \geq 0$ and $d \geq 0$. There exists a problem definable in the fragment of $\mathrm{HGL}_{r+1}(c,d)$ where $\Diamond^+$ and $\Box^+$ are disallowed that is not definable in $\mathrm{HGL}_r(c,d)$.

4.2 Variable numbers of propositional symbols and nominals

We now consider letting the number of propositional symbols or nominals vary whilst keeping the quantifier-rank fixed. For $m \geq 1$, define the digraph $\mathcal{H}_m$ as follows:

- the vertices of $\mathcal{H}_m$ are $\{s, t_1, t_2, \ldots, t_m\}$
- the edges of $\mathcal{H}_m$ are those edges of $\{(s,t_j) : 1 \leq j \leq m\}$.

Hence, $\mathcal{H}_m$ is a star with central vertex s and where all edges are directed away from s to m outer vertices, and can be visualized in Fig. 5.

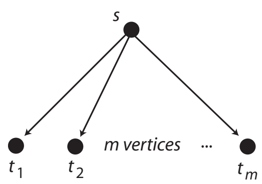

Figure 5. The digraph $\mathcal{H}_m$.

Lemma 26 Let $r \geq 1$, $c \geq 0$ and $d \geq 0$. Define $m = d+2^c$. We have that $\mathcal{H}_m \equiv_{\mathrm{HGL}_r(c,d)} \mathcal{H}_{m+1}$, but that if $c' \geq c$, $d' \geq d$ and $c'+d' = c+d+1$ then there exists a formula φ of $\mathrm{HGL}_r(c',d')$ with the property that $\mathcal{H}_m \models \varphi$ and $\mathcal{H}_{m+1} \not\models \varphi$. Thus, $\mathrm{HGL}_r(c,d) \subset \mathrm{HGL}_r(c',d')$.

Proof As before, we denote vertices of $\mathcal{H}_m$ or $\mathcal{H}_{m+1}$ by using superscripts, as in $s^{\mathcal{H}_{m+1}}$ or $t_1^{\mathcal{H}_m}$, for example.

Let P be a set of c propositional symbols and let N be a set of d nominals. Consider a play of the r-round colouring HGL(P,N)-game on $\mathcal{H}_m$ and $\mathcal{H}_{m+1}$. Suppose that Spoiler makes a colouring-start move so as to build the pointed Kripke $\mathsf{P}\cup\mathsf{N}$-structure $\langle\langle\mathcal{H}_{m+1},\lambda\rangle,v\rangle$. We talk of the *colour* of a vertex x of $\langle\mathcal{H}_{m+1},\lambda\rangle$ as being the set of propositional symbols p of P for which $x\in\lambda(p)$ in union with the set of nominals n of N for which $x=\lambda(n)$ (and do likewise in other Kripke structures). In particular, there are two outer vertices of $\langle\mathcal{H}_{m+1},\lambda\rangle$, say $t_i^{\mathcal{H}_{m+1}}$ and $t_j^{\mathcal{H}_{m+1}}$, with identical colours; moreover, w.l.o.g. we may assume that v is not $t_i^{\mathcal{H}_{m+1}}$. Duplicator replies and builds the pointed Kripke $\mathsf{P}\cup\mathsf{N}$-structure $\langle\langle\mathcal{H}_m,\mu\rangle,u\rangle$ by taking a copy of $\langle\langle\mathcal{H}_{m+1},\lambda\rangle,v\rangle$ and deleting the vertex $t_j^{\mathcal{H}_m}$ (and its incident edge), as well as renaming v as u. Duplicator trivially has a winning strategy in the subsequent r-round HGL(P,N)-game on $\langle\langle\mathcal{H}_m,\mu\rangle,u\rangle$ and $\langle\langle\mathcal{H}_{m+1},\lambda\rangle,v\rangle$.

Alternatively, if Spoiler makes a colouring-start move so as to build the pointed Kripke $\mathsf{P}\cup\mathsf{N}$-structure $\langle\langle\mathcal{H}_m,\mu\rangle,u\rangle$. Duplicator replies and builds the pointed Kripke $\mathsf{P}\cup\mathsf{N}$-structure $\langle\langle\mathcal{H}_{m+1},\lambda\rangle,v\rangle$ by taking a copy of $\langle\langle\mathcal{H}_m,\mu\rangle,u\rangle$ and extending it with a new vertex $t^{\mathcal{H}_{m+1}}$ and edge $(s^{\mathcal{H}_{m+1}},t^{\mathcal{H}_{m+1}})$ so that the colour of $t^{\mathcal{H}_{m+1}}$ is identical to the colour of some point $t_i^{\mathcal{H}_m}$ of $\mathcal{H}_m$ upon which no nominal sits and that is different from u. The point v is chosen as the point (corresponding to) u. Duplicator trivially has a winning strategy in the subsequent r-round HGL(P,N)-game on $\langle\langle\mathcal{H}_m,\mu\rangle,u\rangle$ and $\langle\langle\mathcal{H}_{m+1},\lambda\rangle,v\rangle$. Hence, $\mathcal{H}_m\equiv_{\mathrm{HGL}_r(c,d)}\mathcal{H}_{m+1}$ by Theorem 16.

Let P' be a set of c' propositional symbols and let N' be a set of d' nominals (with c' and d' as in the statement of the result). Consider a play of the r-round colouring HGL$(\mathsf{P}',\mathsf{N}')$-game on $\mathcal{H}_m$ and $\mathcal{H}_{m+1}$. Suppose that Spoiler makes a colouring-start move so as to build the pointed Kripke $\mathsf{P}\cup\mathsf{N}$-structure $\langle\langle\mathcal{H}_{m+1},\lambda\rangle,v\rangle$ where every outer vertex has a different colour and v is the vertex $s^{\mathcal{H}_{m+1}}$ (this is possible irrespective of whether $c'>c$ or $d'>d$). No matter which colouring-start move Duplicator replies with, so as to build $\langle\langle\mathcal{H}_m,\mu\rangle,u\rangle$, Duplicator must choose u as $s^{\mathcal{H}_m}$ (in order to stand a chance of winning the play) and there will exist some outer vertex of $\langle\mathcal{H}_{m+1},\lambda\rangle$ whose colour is not represented in $\langle\mathcal{H}_m,\mu\rangle$. Spoiler clearly wins the subsequent r-round HGL$(\mathsf{P}',\mathsf{N}')$-game on $\langle\langle\mathcal{H}_m,\mu\rangle,u\rangle$ and $\langle\langle\mathcal{H}_{m+1},\lambda\rangle,v\rangle$ by making the appropriate $\Box$-move. The result follows by Theorem 16. $\Box$

Finally, we consider the situation for the logics $\mathrm{HGL}_0(c,d)$, where $c \geq 0$ and $d \geq 0$. Formulae of these logics do not involve the operators $\Diamond$, $\Diamond^+$, $\Box$, $\Box^+$ and $@_n$; that is, they are simply Boolean combinations of propositional symbols and nominals (and $\top$ and $\bot$). Let φ be a formula of one of these logics. It is not difficult to see that:

- if $\mathcal{G}$ is a digraph with at least 2 vertices then $\mathcal{G} \models \varphi$ if, and only if, when we regard all propositional symbols and nominals in φ as Boolean variables, φ is a tautology
- if $\mathcal{G}$ is a digraph with 1 vertex then $\mathcal{G} \models \varphi$ if, and only if, when we regard all propositional symbols and nominals in φ as Boolean variables and make all Boolean variables corresponding to nominals true, the resulting formula φ is a tautology.

In particular, if φ is a formula of one of the above logics:

- if there exists a digraph $\mathcal{G}$ with at least 2 vertices such that $\mathcal{G} \models \varphi$ then the problem defined by φ consists of all digraphs
- if φ is not valid in any digraph with at least 2 vertices and no nominals appear in φ then the problem defined by φ is the empty problem
- if φ is not valid in any digraph with at least 2 vertices and at least 1 nominal appears in φ then either φ is valid in both digraphs with 1 vertex or neither (and both situations are possible).

Thus, we obtain the following result.

Lemma 27 Let $c \geq 0$ and $d \geq 0$.

- For each $c \geq 0$, $\mathrm{HGL}_0(0,0) = \mathrm{HGL}_0(c,0)$, with the class of problems so defined consisting of the two problems consisting of all digraphs and of no digraphs.
- If $d \geq 1$, $\mathrm{HGL}_0(c,d)$ consists of three problems, namely the problems consisting of all digraphs, of no digraphs and of both the digraphs with 1 vertex.

5 Conclusion

Whilst we have obtained a complete classification of the relative expressibilities of the fragments of Hybrid Graph Logic obtain by restricting the quantifier-rank, the number of propositional symbols and the number of nominals, there are some obvious directions for further research. In particular, similar hierarchy results for more expressive hybrid logics, particularly those involving the binder $\downarrow$, should be sought out, and an attempt should be made to extend our hierarchy results to undirected graphs.

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