Variational networks of cube-connected cycles are recursive cubes of rings

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Abstract

In this short note we show that the interconnection networks known as variational networks of cube-connected cycles form a sub-class of the recursive cubes of rings.

1 Introduction

An *interconnection network* is an abstraction of the communications network of a parallel or distributed system as a graph. More precisely, interconnection networks come in families, together with generic routing algorithms, in order to support scalability; that is, the facility to move to a larger network of the same type and but so as to retain the same fundamental routing algorithm. The hypercubes and their dimension-order routing algorithm form the archetypal example of a family of interconnection networks (see, e.g., [4]).

Numerous interconnection networks have been proposed over the last fifty years or so and new designs continue to emerge. There are various reasons for this ongoing investigation and these include the following: the changing face of parallel and distributed systems, which encompass networks-on-chips, supercomputers, clusters and data centre networks, along with new and unforeseen applications, imposes new demands on the underlying interconnection networks (see, e.g., [14]); interconnection networks also feature in peer-to-peer and overlay networks (see, e.g., [15]), social networks (see, e.g., [16]) and wireless sensor networks (see, e.g., [2]); and the 'structured' graphs into which interconnection networks sit feature in combinatorial chemistry (see, e.g., [1]), coding theory (see, e.g., [3]), mathematical physics (see, e.g., [12]) and discrete mathematics in general (often purely as interesting combinatorial objects as regards the latter instantiation; see, e.g., [6, 8, 18]).

As illustrations of recently proposed interconnection networks, we point to the *recursive cubes of rings* (see [10], building on [13]) and the *variational networks of cube-connected cycles* [17]. The purpose of this short note is to point out that the variational networks of cube-connected cycles are, in fact, isomorphic to particular recursive cubes of rings; consequently, the algorithm to find a shortest path joining any two nodes of a variational network of cube-connected cycles and the derivation of the diameter of the network in [17] are actually subsumed by the optimal routing algorithm and derivations in [10].

In the next section, we outline basic concepts and notation and give brief definitions of the recursive cubes of rings and the variational networks of cube-connected cycles. In Section 3, we prove our main result, namely that the variational networks of cube-connected cycles are isomorphic to particular recursive cubes of rings.

2 Basic definitions

We refer the reader to [5, 8, 18] for basic notions relating to interconnection networks and group theory; we only detail here notions and notation that is core to what follows.

For any integer $n \geq 1$, $\mathbb{Z}_n$ denotes the group whose set of elements is $\{0, 1, \ldots, n-1\}$ and whose group multiplication is addition modulo n. For $r \geq 1$, we denote the bit-strings of $\{0,1\}^r$ by **bold type** and we write an r-bit string $\mathbf{x}$ as a tuple of its components $(x_1, x_2, \ldots, x_r)$. Alternatively, we can think of a bit-string of $\{0,1\}^r$ as an element of the group $(\mathbb{Z}_2)^r$. For any $x \in \mathbb{Z}_2$, we write $\bar{x} = x + 1$ (of course, addition is modulo 2). We denote the element of $(\mathbb{Z}_2)^r$ where every component is 0 as $\mathbf{0}$ and for any $1 \leq i \leq r$, we denote the element of $(\mathbb{Z}_2)^r$ with 1 in the ith component and 0 elsewhere as $\mathbf{e}_i$.

We deal with core group theory first, in particular the notions of a semidirect product and a Cayley graph. Let $\mathcal{G}$ be a group and let Ω be a set. Suppose that for every $g \in \mathcal{G}$, there is a permutation φ_g of Ω so that: $\varphi_{1_\mathcal{G}}$ is the identity permutation, where $1_\mathcal{G}$ is the identity of $\mathcal{G}$; and $\varphi_{gh} = \varphi_g \varphi_h$. Then we say that $\mathcal{G}$ *acts* on Ω or that the permutations $\varphi = \{\varphi_g : g \in \mathcal{G}\}$ form an *action* of $\mathcal{G}$ on Ω. Suppose that Ω consists of the elements of a group $\mathcal{Q}$. We say that $\mathcal{G}$ *acts* on $\mathcal{Q}$ if $\mathcal{G}$ acts on $\mathcal{Q}$ as a set and also the action respects the group structure of $\mathcal{Q}$; that is, for each $g \in \mathcal{G}$ and $q_1, q_2 \in \mathcal{Q}$, we have that $\varphi_g(q_1 q_2) = \varphi_g(q_1)\varphi_g(q_2)$ (that is, φ_g is an automorphism of $\mathcal{Q}$). Let $\mathcal{G}$ and $\mathcal{Q}$ be groups so that $\mathcal{G}$ acts on $\mathcal{Q}$ via the action φ. The *semidirect product* $\mathcal{Q} \rtimes \mathcal{G}$ is the group whose element set is $\mathcal{Q} \times \mathcal{G}$ and where the group multiplication is defined via $(q, g)(q', g') = (q\varphi_g(q'), gg')$, for all $g, g' \in \mathcal{G}$ and $q, q' \in \mathcal{Q}$.

A *Cayley graph* $Cay(\mathcal{G}; \Gamma_\mathcal{G})$ is an undirected graph obtained from a finite group $\mathcal{G}$ and a generating set $\Gamma_\mathcal{G}$, where $\Gamma_\mathcal{G}$ is closed under inverses and does not contain the identity element: the vertex set of $Cay(\mathcal{G}; \Gamma_\mathcal{G})$ is the set of elements of $\mathcal{G}$; and there is an edge (g_1, g_2) if, and only if, $g_2 = g_1\gamma$, for some $\gamma \in \Gamma_\mathcal{G}$. A graph is *vertex-transitive* if for any distinct vertices u and v, there is an automorphism of the graph mapping u to v. Vertex-transitivity is an important property as, for one thing, when we have an interconnection network whose underlying communication graph is vertex-transitive, we can deploy the same routing algorithm at every processor. Cayley graphs feature strongly as interconnection networks as every Cayley graph is necessarily vertex-transitive (see, e.g., [7, 9] for more on the role of Cayley graphs as interconnection networks).

Recursive cubes of rings originated in [13] but, as defined there, are not necessarily Cayley graphs. Recently, Mokhtar and Zhou [10] refined recursive cubes of rings so as to force them to be Cayley graphs and it is their definition that we give below. Henceforth, we refer to our structures as 'graphs' rather than 'interconnection networks' as we are not concerned here with routing algorithms or any other features of interconnection networks.

Let $n \geq 2$, $n \geq d \geq 1$ and $r \geq 3$ so that $dr = 0 \pmod{n}$. Let the group $\mathbb{Z}_r$ act on the group $(\mathbb{Z}_2)^n$ as follows: for $p \in \mathbb{Z}_r$, the map φ_p cyclically shifts the components of $(x_1, x_2, \ldots, x_n)$ by $pd \pmod{n}$ components to the right; so, for example, if $n = 5$, $d = 2$, $r = 10$ and $p = 3$ then $pd \pmod{n} = 1$ and $\varphi_3((x_1, x_2, x_3, x_4, x_5)) = (x_5, x_1, x_2, x_3, x_4)$. Note that our condition that $dr = 0 \pmod{n}$ ensures that a group action results and we call this action φ.

Definition 1 Form the semidirect product $(\mathbb{Z}_2)^n \rtimes \mathbb{Z}_r$ w.r.t. the action φ. The *recursive cubes of rings* $Q_n(d, r)$ is the Cayley graph of $(\mathbb{Z}_2)^n \rtimes \mathbb{Z}_r$ where the (inverse-closed) set of generators is $\{(\mathbf{0}, 1), (\mathbf{0}, r-1), (\mathbf{e}_1, 0), (\mathbf{e}_2, 0), \ldots, (\mathbf{e}_d, 0)\}$.

Combinatorially, $Q_n(d, r)$ can be constructed as follows:

- take 2^n copies of a cycle of length r with each copy labelled by an element of $(\mathbb{Z}_2)^n$
- for each $0 \leq i \leq r-1$, include an edge joining two vertices named i in different cycles if, and only if,
  - the labels of the cycles within which they lie differ in exactly one component, and
  - this component lies in $\{id + j + 1 \pmod{n} : j = 0, 1, \ldots, d-1\}$.

Consequently, the graph $Q_n(d, r)$ is the Cartesian product $Q_n \times C_r$, where Q_n is an n-dimensional hypercube and C_r is the cycle of length r but with some edges removed (as dictated by the action φ). (Note that one point of defining interconnection networks in this way is so that the 'overall structure' of the Cartesian product is retained but the degree is significantly reduced, as low degree interconnection networks are beneficial from a practical perspective; moreover, the group theory leads to succinct descriptions and provides a framework for routing.) The graphs $Q_3(2, 3)$ and $Q_3(1, 3)$ can be visualized as in Fig. 1 of [10].

In order to get an appreciation of the edges of a recursive cubes of rings, let us consider $Q_n(d, r)$ where $n = 8$, $d = 3$ and $r = 8$. A vertex 0 of some cycle of length 8 in $Q_8(3, 8)$ that is labelled $\mathbf{x} \in \{0, 1\}^8$ is adjacent to the vertex 0 in the 3 cycles whose labels are obtained from $\mathbf{x}$ by individually flipping each of the bits x_1, x_2, x_3; a vertex 1 of some cycle of length 8 in $Q_8(3, 8)$ that is labelled $\mathbf{x} \in \{0, 1\}^8$ is adjacent to the vertex 1 in the 3 cycles whose labels are obtained from $\mathbf{x}$ by individually flipping each of the bits x_4, x_5, x_6; a vertex 2 of some cycle of length 8 in $Q_8(3, 8)$ that is labelled $\mathbf{x} \in \{0, 1\}^8$ is adjacent to the vertex

2 in the 3 cycles whose labels are obtained from $\mathbf{x}$ by individually flipping each of the bits x_7, x_8, x_1; and so on.

The variational networks of cube-connected cycles were recently proposed in [17] as extensions of the well-known cube-connected cycles from [11]. Before we define these graphs, let us define two automorphisms of the group $(\mathbb{Z}_2)^r$:

- the *left-automorphism* ρ_l acts via

$$\rho_l : (x_1, x_2, \ldots, x_{r-1}, x_r) \mapsto (x_2, x_3, \ldots, x_r, x_1)$$

- the *right-automorphism* ρ_r acts via

$$\rho_r : (x_1, x_2, \ldots, x_{r-1}, x_r) \mapsto (x_r, x_1, \ldots, x_{r-2}, x_{r-1})$$

(note that ρ_l and ρ_r are indeed group automorphisms).

Definition 2 The *variational network of cube-connected cycles $RVCCC_r$*, for $r \geq 2$, has:

- vertex set $(\mathbb{Z}_2)^r \times \mathbb{Z}_{2r}$, and
- the edges are of three types:

  (A) $((\mathbf{x}, p), (\rho_l(\mathbf{x}), p+1))$, for all $\mathbf{x} \in (\mathbb{Z}_2)^r$ and $p \in \mathbb{Z}_{2r}$

  (B) $((\mathbf{x}, p), (\rho_r(\mathbf{x}), p-1))$, for all $\mathbf{x} \in (\mathbb{Z}_2)^r$ and $p \in \mathbb{Z}_{2r}$

  (C) $(((\mathbf{x}, x_r), p), ((\mathbf{x}, \bar{x}_r), p))$, for all $\mathbf{x} \in (\mathbb{Z}_2)^{r-1}$, $x_r \in \mathbb{Z}_2$ and $p \in \mathbb{Z}_{2r}$.

The graph $RVCCC_3$ can be visualized as in Fig. 2 of [17]. Looking at Definitions 1 and 2, it might appear that recursive cubes of rings and variational networks of cube-connected cycles are structurally very different. However, we now prove that this is not the case.

3 The main result

Define the map ψ on the vertices of $RVCCC_r$ as follows: for each $(\mathbf{x}, p) \in (\mathbb{Z}_2)^r \times \mathbb{Z}_{2r}$, the vertex $(\mathbf{x}, p)$ is mapped to the vertex $\psi((\mathbf{x}, p)) = (\rho_r^{p+1}(\mathbf{x}), p)$.

Lemma 3 The map ψ is a bijection.

Proof Suppose that $\psi((\mathbf{x}, p)) = \psi((\mathbf{y}, q))$, for two distinct vertices $(\mathbf{x}, p)$ and $(\mathbf{y}, q)$ of $RVCCC_r$; so, $(\rho_r^{p+1}(\mathbf{x}), p) = (\rho_r^{q+1}(\mathbf{y}), q)$. Hence, $p = q$ and $\rho_r^{p+1}(\mathbf{x}) = \rho_r^{p+1}(\mathbf{y})$, with $\mathbf{x} = \mathbf{y}$. So, ψ is a bijection. $\square$

Proposition 4 The variational network of cube-connected cycles $RVCCC_r$ is isomorphic to the graph $\mathcal{G}$ whose:

- vertex set is $(\mathbb{Z}_2)^r \times \mathbb{Z}_{2r}$, and

- whose edges are of one of three types:

  (1) $((\mathbf{x},p),(\mathbf{x},p+1))$, for all $\mathbf{x}\in(\mathbb{Z}_2)^r$ and $p\in\mathbb{Z}_{2r}$

  (2) $((\mathbf{x},p),(\mathbf{x},p-1))$, for all $\mathbf{x}\in(\mathbb{Z}_2)^r$ and $p\in\mathbb{Z}_{2r}$

  (3) $(((x_1,\ldots,x_q,x_{q+1},x_{q+2},\ldots,x_r),p),$
  $((x_1,\ldots,x_q,\bar{x}_{q+1},x_{q+2},\ldots,x_r),p)),$
  for all $(x_1,x_2,\ldots,x_r)\in(\mathbb{Z}_2)^r$ and $p\in\mathbb{Z}_{2r}$, where $q=p \pmod r$.

Proof Consider an edge of $RVCCC_r$ of type (A), with reference to Definition 2, namely $((\mathbf{x},p),(\rho_l(\mathbf{x}),p+1))$, for some $\mathbf{x}\in(\mathbb{Z}_2)^r$ and $p\in\mathbb{Z}_{2r}$. Under the map ψ this edge is mapped to the pair

$$((\rho_r^{p+1}(\mathbf{x}),p),(\rho_r^{p+2}\rho_l(\mathbf{x}),p+1))=((\rho_r^{p+1}(\mathbf{x}),p),(\rho_r^{p+1}(\mathbf{x}),p+1)).$$

Consequently, the type (A) edge $((\rho_l^{p+1}(\mathbf{x}),p),(\rho_l(\rho_l^{p+1}(\mathbf{x})),p+1))$ of $RVCCC_r$ is mapped to the type (1) edge $((\mathbf{x},p),(\mathbf{x},p+1))$ of $\mathcal{G}$.

Consider an edge of $RVCCC_r$ of type (B), namely $((\mathbf{x},p),(\rho_r(\mathbf{x}),p-1))$, for some $\mathbf{x}\in(\mathbb{Z}_2)^r$ and $p\in\mathbb{Z}_{2r}$. Under the map ψ, this edge is mapped to the pair

$$((\rho_r^{p+1}(\mathbf{x}),p),(\rho_r^{p}\rho_r(\mathbf{x}),p-1))=((\rho_r^{p+1}(\mathbf{x}),p),(\rho_r^{p+1}(\mathbf{x}),p-1)).$$

Consequently, the type (B) edge $((\rho_l^{p+1}(\mathbf{x}),p),(\rho_r(\rho_l^{p+1}(\mathbf{x})),p-1))$ of $RVCCC_r$ is mapped to the type (2) edge $((\mathbf{x},p),(\mathbf{x},p-1))$ of $\mathcal{G}$.

Consider an edge of $RVCCC_r$ of type (C), namely $(((\mathbf{x},x_r),p),((\mathbf{x},\bar{x}_r),p))$, for some $\mathbf{x}\in(\mathbb{Z}_2)^{r-1}$, $x_r\in\mathbb{Z}_2$ and $p\in\mathbb{Z}_{2r}$. Under the map ψ, this edge is mapped to the pair $((\rho_r^{p+1}((\mathbf{x},x_r)),p),(\rho_r^{p+1}((\mathbf{x},\bar{x}_r)),p))$

$$=(((x_{r-q},\ldots,x_{r-1},x_r,x_1,\ldots,x_{r-q-1}),p),$$
$$((x_{r-q},\ldots,x_{r-1},\bar{x}_r,x_1,\ldots,x_{r-q-1}),p)),$$

where $q=p \pmod r$ and if a subscript i of any x_i evaluates to 0, then we replace it with r. So, an edge of $RVCCC_r$ of type (C) is mapped to an edge of $\mathcal{G}$ of type (3). The result follows as ψ is a bijection (by Lemma 3). □

However, the graph $\mathcal{G}$ from Proposition 4 is nothing else than the recursive cubes of rings $Q_r(1,2r)$. Hence, we have proven the following result.

Theorem 5 For $r\geq 2$, the variational network of cube-connected cycles $RVCCC_r$ is isomorphic to the recursive cubes of rings $Q_r(1,2r)$.

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